Algebra II 2023 Midterm Solutions

1. Answer 'T(rue)' or 'F(alse)'.

   (a) Multiplication in a field is commutative. T

   (b) Every integral domain of characteristic 0 is infinite T - The characteristic of a ring divides its order, so rings of finite characteristic must be finite.

   (c) The product of two non-units of a ring can be a unit. F- If ab is a unit, so $ab \cdot c = 1$ for some c, then $a \cdot bc = 1$ so a is a unit.

   (d) A polynomial in $F[x]$ of degree n can have at most n zeros in any extension E of F. T

   (e) The units of $F[x]$ are exactly the non-zero elements of a field F. T

   (f) $\mathbb{Q}$ is an ideal of $\mathbb{R}$. F

   (g) A ring with zero divisors may contain a prime fields as a subring. T- $\mathbb{Z} \times \mathbb{Z}$ is such a ring.

   (h) $\mathbb{Q}$ is an extension of $\mathbb{Z}_2$ F - wrong characteristic

   (i) Every non-constant polynomial in $F[x]$ has a zero in every extension of F. F- not every extension

	T or F	Comment/Reason
a		
b		
c		
d		
e		
f		
g		
h		
i		

2. The following Short Answer and Computations questions can be answered without work/explanation. (If you are unsure of your answer though, a short explanation might get part marks.)

   (a) What is the characteristic of $\mathbb{Z}_3 \times \mathbb{Z}$? 0

   (b) Expand the following into a single non-factored polynomial in $\mathbb{Z}_6[x]$:

$$(2x^2 + 3x + 4)(3x^2 + 2x + 3).$$

$$x^4(6) + x^3(4+9) + x^2(12+6) + x(8+9) + 12) = \boxed{x^3 - x}$$

   (c) Find all ideals of $\mathbb{Z}_{12}$. $\{0\}$ and $\langle d \rangle$ for the divisors $d = 1, 2, 3, 4$ and 6 of 12.

3. The following 'Concepts' from the text are definitions that often need slight modification to make them fully correct. If they are correct, write correct, otherwise, fix them so they are correct.

   (a) If $ab = 0$ then a and b are *zero divisors*. ...and neither are 0, then ...

   (b) An element of $R[x]$ is a infinite formal sum

$$a_0 + a_1 x + a_2 x^2 + \ldots$$

   where $a_i \in R$ for $i \in \mathbb{N}$. ... and $a_i = 0$ for all but finitely many i.

   (c) A maximal ideal of a ring R is an ideal that is not contained in another ideal of R. ...not contained in another proper ideal of R.

4. Find the order of the matrix ring $M_2(\mathbb{Z}_2)$, count or list the units of the ring.

Solution

There are 4 entries in a 2×2 matrics, and two choices of element per entry, so the ring has order $\boxed{2^4}$. A unit is an invertible element, as $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is clearly the identity, this means the invertible matrices. A matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ if and only if its determinant $ac - bd$ is non-zero. For this to happen we need exactly one of

$$a = c = 1 \qquad \text{or} \qquad b = d = 1$$

to hold. There are 3 matrices in which the first holds and the second doesn't, and 3 in which the second does and the first doesn't. So there are six units

$$\boxed{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]}$$

5. (a) What is an integral domain?

   (b) Give an example of an integral domain that isn't a field.

   (c) Show that the characteristic of a subdomain of an integral domain D is the same as the characteristic of D.

Solution

i) A commutative ring with unity having no zero-divisors. ii) $\mathbb{Z}$. iii) The characteristic of a domain is the characteristic of the ring $\langle 1 \rangle$ generated by its unit 1. A subdomain must contain the unit and so all of $\langle 1 \rangle$ so has the same characteristic.

6. State and prove Euler's Theorem.

Solution

See text.

7. Show that for prime p the polynomial $x^p + a$ in $\mathbb{Z}_p[x]$ is not irreducible for any $a \in \mathbb{Z}_p$.

Solution

It has zero $-a$ as $-a^p + a \equiv_p -a + a = 0$, so by the factor theorem $(x+a)$ divides it.

8. Show for a ring homomorphism $\phi : R \to R'$, and an ideal N' of R', that $N = \phi^{-1}[N']$ is an ideal. (You may use that it is a subring of R'.)

Solution

For $r \in R$ and $n \in N$ we have that $\phi(r) \in R'$ and $\phi(n) \in N'$ so as N' is an ideal $\phi(rn) = \phi(r)\phi(n) \in N'$. So $rn \in \phi^{-1}[N'] = N$, as needed to show that N is an ideal.

9. Show that for a maximal ideal M of a commutative ring R with unity, the quotient ring R/M is a field.

Solution

As M is maximal, it is not all of R, so the commutative ring R/M also has unity. If there is some non-zero element $a + R/M$ without an inverse, then the ideal $\langle a + R/M \rangle$ does not contain $1 + R/M$, so is a non-trivial proper ideal of R/M. Its pre-image $\gamma^{-1}(\langle a + R/M \rangle)$ under the canonical quotient $\gamma : R \to R/M$ is an ideal(by the previous question) which contains $M \cup \{a\}$ but does not contain 1 is a proper ideal of R that properly contains M. This contradicts the fact that M is maximal. Thus all elements of R/M are invertible, so it is a field.

10. Find the irreducible polynomial of $\alpha = \sqrt{3 - \sqrt{6}}$ over $\mathbb{Q}$. Prove that it is irreducible.

Solution

Setting $x = \sqrt{3-\sqrt{6}}$ we see that $x^2 = 3-\sqrt{6}$ so $6 = (x^2-3)^2 = x^4-6x^2+9$. So α is a zero of $f(x) = x^4 - 6x^2 + 3$. Checking that $\pm 1, \pm 3$ are not zeros, we see that $f(x)$ has no linear factors. So if it is reducible over $\mathbb{Z}$ it is

$$\begin{aligned} x^4 - 6x^2 + 3 &= (x^2 + ax + b)(x^2 + cx + d) \\ &= x^4 + (a+b)x^3 + (ac + b + d)x^2 + (ad + cb)x + bd. \end{aligned}$$

Equating coefficients we get that $b = -a$; which with $ad + cb = 0$ gives us that $a = b = 0$ or $d = b$. We cannot have $d = b$ as then $b = \sqrt{3}$; and we cannot have $a = b = 0$ as then $0d = 3$. Thus $f(x)$ is irredicible over $\mathbb{Z}$. and so over $\mathbb{Q}$.