

# Combinatorics 2022 Midterm Solutions

1. We are playing the game of Nim. (There are piles of stones; two players take turns removing stones from a single pile, and win by taking the last stone.)

- (a) Show that the position  $(21, 14, 7)$  a winning (unbalanced) position?
- (b) Give a move that Player 1 could play to leave Player 2 in a losing position.

## Solution

Looking at the binary expansions of the pile sizes,

	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$
21 :	1	0	1	0	1
14 :	0	1	1	1	0
7 :	0	0	1	1	1

we see that there are odd columns— columns that sum to odd numbers— so it is a winning position. Player  $A$  should leave only even columns, so must play the pile with 21 stones, as it is the only row with a stone in the largest odd column. Removing this row, the odd columns are the  $2^3$  and the  $2^0$  columns, so she should leave  $2^3 + 2^0 = 9$  stones in the pile. To do so she takes 12 of the 21 stones from that pile.

2. Count the number of...

- (a) 11-permutations of  $\{3 \cdot a, 3 \cdot b, 3 \cdot c, 3 \cdot d\}$ . Choose the missing number in 4 ways, and then order a multiset:  $4 \frac{11!}{3!3!3!2}$ . Alternately, observe that there is a one-to-one correspondence between the permutations of this multiset and the 11-permutations: just remove the last element. So there are  $\frac{12!}{(3!)^4}$  of them.
- (b)  $r$ -combinations of  $\{1 \cdot a_1, 2 \cdot a_2, \infty \cdot a_3, \dots, \infty \cdot a_k\}$ . We can divide it into cases depending on how many elements of  $\{a_1, a_2, a_2\}$  are in the combination:

$$\binom{r+k-3}{k-3} + 2 \binom{r+k-4}{k-3} + 2 \binom{r+k-5}{k-3} + \binom{r+k-6}{k-3}.$$

Alternately, we can use PIE and count the non-negative integer solutions to  $x_1 + \dots + x_k = r$  with  $x_1 \leq 1$  and  $x_2 \leq 2$ . Letting  $s = \binom{r+k-1}{k-1}$  be the number of solutions to  $x_1 + \dots + x_k = r$  we let  $a_1$  count those with  $x_1 \geq 2$ , let  $a_2$  count those with  $x_2 \geq 3$  and let  $a_{12}$  count those satisfying both of these properties. Then the number of good solutions is  $s - a_1 - a_2 + a_{12} =$

$$\binom{r+k-1}{k-1} - \binom{r+k-3}{k-1} - \binom{r+k-4}{k-1} + \binom{r+k-6}{k-1}.$$

(c) ways 10 people, consisting of 5 couples, can be arranged in a circle so that each person is diametrically opposite their partner. Fixing the first person, there are  $4!$  orderings of the couples, and then once the couples are ordered, there are  $2^4$  arrangements within the couples (omitting the one with the fixed first person), so there are  $4!2^4$  ways to arrange them.

3. Give the probability of getting a three-of-a-kind in a randomly dealt hand of 5 cards from a standard 52 card deck. ( For each choice of one of 13 different 'marks' and four different 'suits', there is one card in the deck with that mark and suit. A hand of five cards is a three-of-a-kind if it contains three cards of one mark. None of the other two cards can have that same mark, nor can the two of them share a mark.)

**Solution**

There are 13 choices for the mark, and 4 choices for the three cards of that make in the three-of-a-kind. To get the other two cards, we can't get the same mark, and we cannot get a pair of another mark, or we have a better hand, so we have to choose the two marks in  $12 \cdot 11$  ways, and the suits in  $4^2$  ways. The probability is therefore  $\frac{13 \cdot 12 \cdot 11 \cdot 4^3}{\binom{52}{5}}$ .

4. Show that the ramsey number  $R(3, 3)$  is 6.

**Solution**

You can show that it is greater than 5 with a nice two-colouring of  $K_5$ . Argue as follows to show that it is at most 6.

Consider the colours on the five edges incident to a given vertex  $v$ . By the pigeonhole principle, there is some set of three edges that have the same colour. Assume, wlog, these three edges are all blue, and let  $x, y$  and  $z$  be the neighbours of  $v$  via these edges. If any two of  $x, y$  and  $z$  induce a blue edge, then with  $v$  we have a blue  $K_3$ . If no two of them do, then  $x, y$  and  $z$  induces a red  $K_3$ .

5. Consider the ordering of the permutations of  $[n]$  that we constructed recursively in class. (We list the permutations of  $[n]$  by repeating each permutation of  $[n - 1]$  (in order)  $n$  times, and then move  $n$  through the copies of each permutation. We augmented this by putting arrows on the top of each element of the permutation. ) With respect to this ordering of the 720 permutations of  $[6]$ :

- (a) What is the  $427^{th}$  permutation?
- (b) What are its arrows?
- (c) What is the next permutation?

**Solution**

At 427 the 6 is in its  $427 \bmod 12 = 7^{th}$  position where its positions are

$$\begin{array}{cccccc} 6 & 5 & 4 & 3 & 2 & 1 \\ 7 & 8 & 9 & 10 & 11 & 12 \end{array}$$

In position 7 it has a right arrow. So we have

$$\overrightarrow{6} - - - - - .$$

Removing 6 we are in the  $\lceil 427 \cdot 1/6 \rceil = 72^{nd}$  permutation of  $[5]$ . The 5 is in position  $72 \bmod 10 = 2$  of

$$\begin{array}{cccccc} 5 & 4 & 3 & 2 & 1 \\ 6 & 7 & 8 & 9 & 10 \end{array}$$

So we have

$$\overrightarrow{6} - - - \overleftarrow{5} - .$$

The 4 is in position  $\lceil 72 \cdot 1/5 \rceil = 15$  and mod 8 this is 7:

$$\overrightarrow{6} - - \overrightarrow{4} \overleftarrow{5} - .$$

The 3 is in position  $\lceil 15 \cdot 1/4 \rceil = 4$  which mod 6 is 4:

$$\overrightarrow{6} \overrightarrow{3} - \overrightarrow{4} \overleftarrow{5} - .$$

The 2 is in position  $\lceil 4 \cdot 1/3 \rceil = 2$ :

$$\overrightarrow{6} \overrightarrow{3} \overleftarrow{2} \overrightarrow{4} \overleftarrow{5} \overrightarrow{1} .$$

This answers parts (a) and (b). To find the next permutation, observe that 6 is mobile, so we get

$$\overrightarrow{3} \overrightarrow{6} \overleftarrow{2} \overrightarrow{4} \overleftarrow{5} \overrightarrow{1} .$$

6. How many permutations of  $[n]$  have 3 inversions?

**Solution**

The inversion sequence  $x_1, \dots, x_n$  of a permutation is a non-negative integer solution to  $x_1 + x_2 + \dots + x_n = 3$  with  $x_n = 0$ ,  $x_{n-1} \leq 1$  and  $x_{n-2} \leq 2$ . We throw out the  $x_n$  and count that there are  $a_0 = \binom{n-2+3}{3} = \binom{n+1}{3}$  solutions to

$$x_1 + x_2 + \dots + x_{n-1} = 3.$$

Of these  $a_1 = n - 1$  violate  $x_{n-1} \leq 1$ ,  $a_2 = 1$  violate  $x_{n-2} \leq 2$  and  $a_{12} = 0$  violate both of these. So there are  $a_0 - a_1 - a_2 + a_{12} = \binom{n+1}{3} - n$  permutations of  $[n]$  with 3 inversions.

7. Prove that, for all real numbers  $r$  and all integers  $k$  and  $m$ ,

$$\binom{r}{m} \binom{m}{k} = \binom{r}{k} \binom{r-k}{m-k}.$$

## Solution

Because  $r$  can be any real we must use the following definition of  $\binom{r}{m}$ :

$$\binom{r}{m} = \begin{cases} \frac{r(r-1)\cdots(r-m+1)}{m!} & \text{if } m \geq 1 \\ 1 & \text{if } m = 0 \\ 0 & \text{if } m \leq -1 \end{cases}.$$

If  $k < 0$ , or  $k > m$ , then we get 0 on both sides of the identity, so it holds. Thus we may assume that  $0 \leq k \leq m$ . If  $k = m$  then  $\binom{m}{k} = 1$  and  $\binom{r-k}{m-k} = \binom{r-k}{0} = 1$  and so

$$\binom{r}{m} \binom{m}{k} = \binom{r}{m} = \binom{r}{k} = \binom{r}{k} \binom{r-k}{m-k}.$$

So we may assume that  $0 \leq k < m$ . In this case we have

$$\begin{aligned} \binom{r}{m} \binom{m}{k} &= \frac{r \cdot r - 1 \cdots r - m + 1}{m!} \cdot \frac{m!}{k!(m-k)!} \\ &= \frac{r \cdot r - 1 \cdots r - k + 1}{k!} \cdot \frac{r - k \cdots r - m + 1}{(m-k)!} \\ &= \binom{r}{k} \cdot \frac{r - k \cdots r - k - (m - k) + 1}{(m-k)!} \\ &= \binom{r}{k} \cdot \binom{r-k}{m-k} \end{aligned}$$