Combinatorics 2022 Final Solutions

1. The Lucas numbers $\ell_0, \ell_1, \ell_2, \dots$ are defined recusively by $\ell_0 = 2, \ell_1 = 1$, and for $n \geq 2$, $\ell_n = \ell_{n-2} + \ell_{n-1}$. Where $f_0, f_1, f_2, \dots$ are the Fibonacci numbers, show that $\ell_n = f_{n-1} + f_{n+1}$ for $n \geq 1$.

Solution

We use induction. Base case: As $\ell_1 = 1 = 0 + 1 = f_0 + f_2$ the identity holds for $n = 1$.
Now let $n > 1$ and assume the identity holds for smaller n. Then

$$\begin{aligned}
\ell_n &= \ell_{n-2} + \ell_{n-1} \\
&= (f_{n-3} + f_{n-1}) + (f_{n-2} + f_n) \\
&= (f_{n-3} + f_{n-2}) + (f_{n-1} + f_n) \\
&= f_{n-1} \qquad\qquad + f_{n+1},
\end{aligned}$$

and the identity holds for n. We used the induction hypothesis in second line and the fibonacci recurrence in the last line.

2. Using an exponential generating function, find the number h_n of ways of colouring the squares of a $1 \times n$ chessboard with 4 colours red, yellow, blue and white, so that there are an even number of red and blue tiles.

Solution

The exponential generating function for $h_0, h_1, h_2, \dots$ is

$$\begin{aligned}
g^{(e)}x &= (1 + x + \frac{x^2}{2!} + \dots)^2 (1 + \frac{x^2}{2!} + \dots)^2 \\
&= e^{2x}\left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{1}{4}(e^{4x} + 2e^{2x} + 1) \\
&= \frac{1}{4}\left((1 + 4x + 4^2\frac{x^2}{2!} + 4^3\frac{x^3}{3!} \dots) + 2(1 + 2x + 2^2\frac{x^2}{2!} + 2^3\frac{x^3}{3!}) + 1\right) \\
&= 1 + x + (2 + 4)\frac{x^2}{2!} + (2^2 + 4^2)\frac{x^3}{3!} + \dots
\end{aligned}$$

From this we read off that $h_n = 2^{n-1} + 4^{n-1}$ for $n \geq 1$. And $h_0 = 1$.

3. Find an expression for $\sum_{i=1}^{n} i^5$. This might help:

0		1		32		243		1024		3125
	1		31		211		781		2101	…
		30		180		570		1320	…	
			150		390		750	…		
				240		360	…			
					120	…				

Solution

As n^5 is a polynomial of degree 5, the next row of the difference table shown is all zeros. Using our formula for an entry h_i of the first row in terms of the 0^{th} diagonal of a difference table, we get that

$$n^5 = 0\binom{n}{0} + 1\binom{n}{1} + 30\binom{n}{2} + 150\binom{n}{3} + 240\binom{n}{4} + 120\binom{n}{5}$$

for all n.
Using the identity $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k}{k}$ we sum the n^5 to get

$$\sum_{i=1}^{n} i^5 = 1\binom{n+1}{2} + 30\binom{n+1}{3} + 150\binom{n+1}{4} + 240\binom{n+1}{5} + 120\binom{n+1}{6}.$$

4. Recall that $l_n(k)$ is the number of partitions of n in which no part is smaller than k.
EITHER: show that $l_n(k) = l_n(k+1) + l_{n-k}(k)$ for all $1 \leq k < n$.
OR: use the following rules to compute $l_7(3)$.

   (a) $l_0(k) = 1$ for all $k \geq 0$.
   (b) $l_n(k) = 0$ if $1 \leq n < k$.
   (c) $l_n(n) = 1$ for all $n \geq 0$.
   (d) $l_n(k) = l_n(k+1) + l_{n-k}(k)$ for all $1 \leq k < n$.

Solution

Consider the family $\mathcal{P}$ of partitions of n in which no part is smaller than k. Let $\mathcal{P}_0$ be the subfamily of those that have a part of size k, and let $\mathcal{P}_1$ be the subfamily of those that have no part of size k. Then $l_n(k) = |\mathcal{P}_0| + |\mathcal{P}_1|$. The family $\mathcal{P}_1$ is the family of partitions of n in which no part is smaller than $k+1$, so we have $|\mathcal{P}_1| = l_n(k+1)$. For each partition in $\mathcal{P}_0$ remove one part of size k, we get exactly the partitions of $[n-k]$ in which no part is smaller than k. So $|\mathcal{P}_0| = l_{n-k}(k)$. Thus $l_n(k) = |\mathcal{P}_0| + |\mathcal{P}_1| = l_n(k+1) + l_{n-k}(k)$, as needed.

OR

We compute

$$\begin{aligned}
l_7(3) &= l_7(4) + l_4(3) \\
&= l_7(5) + l_3(4) + l_4(4) + l_1(3) = l_7(5) + 0 + 1 + 0 \\
&= l_7(6) + l_2(5) + 1 = l_7(6) + 0 + 1 \\
&= l_7(7) + l_1(6) + 1 = 1 + 0 + 1 \\
&= 2
\end{aligned}$$

5. Consider the family $\mathcal{A} = (A_1, A_2, \ldots, A_6)$ of sets where

$$\begin{array}{lll} A_1 = \{1,2\} & A_2 = \{1,2,3\} & A_3 = \{3,4\} \\ A_4 = \{4,5\} & A_5 = \{5,6\} & A_6 = \{6,1\}. \end{array}$$

How many different SDRs does $\mathcal{A}$ have?

Solution

Throwing out the 1 in A_2 you get the problem from the homework. In this case there exactly two SDRs: either choose i from A_i for all i, or choose $i+1$ from A_i for all i. Indeed, to have an SDR one must choose either 1 or 2 from A_1. In the former case one must then choose 6 from A_6, and then 5 from A_5 etc. In the latter case one must then choose 3 from A_2 and then 4 from A_3, etc.
Now, with the 1 in A_2 we have these two SDRs, and in any other SDR, the representative of A_2 is 1. But then we must choose 2 from A_1. Then we must choose 6 from A_6 and so 5 from A_5, etc. In total, there are three SDRs.

6. Recall that a block design $\mathcal{B} = \{B_1, \ldots, B_b\}$ consists of b blocks (subsets of the set $[v]$ of size k), such that any two varieties (elements of $[v]$) occur together in exactly λ blocks. Show that $\frac{\lambda(v-1)}{k-1}$ is an integer r and that $bk = vr$.

Solution

Let $x \in [v]$ be any variety. It must occur with $[v-1]$ other varieties λ times each, and in each of the r blocks that it occurs in, it occurs with $k-1$ other varieties one time. So it must occur in $r = \lambda(v-1)/(k-1)$ blocks, and so this value must be an integer.

For the second identity, we count in two ways the number X of pairs (B, v) of block B and varities vB. For each of b blocks there are k such pairs, so $X = bk$. On the other hand, each of v varieties is in r such pairs, so $X = vr$. Thus $bk = X = vr$.

7. Explain what a latin square of order n is, and what a MOLS of order n is. Show that there cannot exist a MOLS of order n containing n latin squares.

Solution

A latin square of order n is an $n \times n$ matrix over $\mathbb{Z}_n$ such that for any $i \in \mathbb{Z}_n$ the entry i occurs exactly once in each row and column.

Two latin squares $M_\alpha = [m^\alpha_{ij}]$ and $M_\beta = [m^\beta_{ij}]$ are orthogonal if the pairs $(m^\alpha_{i,j}, m^\beta_{i,j})$ for all $i, j \in [n]$ are distinct.

A family of latin squares is a MOLS if any two latin squares in the family are othogonal.

Given such a family $\{M_1, \ldots M_r\}$, we may assume that every latin square in the family is in standard form: $m^\alpha_{0,j} = j$ for all α and j. Consider, therefore the entries

$$m^1_{1,0}, m^2_{1,0}, \ldots, m^r_{1,0} \in \mathbb{Z}_n$$

None of them are 0 as for any α $m^\alpha_{0,0} = 0$ is in the same column. And they must be distinct, as the pair (i, i) already exists as $(m^\alpha_{0,i}, m^\beta_{0,i})$. So r can be at most $n - 1$.

8. Show that every graph on at least 2 vertices has two vertices with the same degree.

Solution

The degrees of the n vertices are between 0 and $n - 1$. If some vertes has degree $n - 1$ then it has edges with every other vertex, and so no vertex can have degree 0. Thus the n degrees are either in $\{0, \ldots, n-2\}$ or $\{1, \ldots, n-1\}$ Either way, two of the degrees are the same, by the pigeonhole principle.

9. Show for every even $n \geq 2$ there is a connected non-hamiltonian graph on n vertices with min degree $n/2 - 1$.

Solution

Take two $K_{n/2}$ and add a single edge between them.