Graduate Graph Theory

KNU Math 848

Classnotes

Mark Siggers

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These notes are for a graduate level introduction to Graph Theory. They draw largely from Diestal's text [2] Graph Theory, West's text [4] Introduction to Graph Theory, and Devos' class notes [1]. The source for Section 10 is Hell and Nešetřil's text Graphs and Homomorphisms [3].

1 Introduction to Graph Theory

Definition 1.1. A graph G consists of a non-empty set $V(G)$ of *vertices* and a set $E(G) \subset 2^{\binom{V}{2}}$ of *edges*: or 2-element subsets of V. Where it is clear, we write V for $V(G)$ and E for $E(G)$.

We often write uv for $\{u, v\}$, and say it is an edge of G of in many ways: u and v are *adjacent*, u and v are *neighbours*, $u \sim v$, u and v are the *endpoints* of (some edge) e.

One can generalise the idea of a graph to

- *digraphs* by making edges ordered sets: uv and vu are different,
- *multigraphs* by making the edgeset E a multiset,
- *hypergraphs* by allowing edges to contain more than two vertices,
- *non-simple graphs* by allowing edges of the form uu.

We mostly don't consider these generalisations.

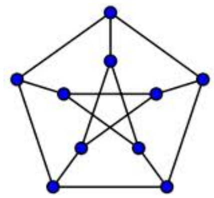
Two graphs G and G' are *isomorphic* if there is a bijection $f : V(G) \to V(G')$ such that f and f^{-1} both preserve adjacency. We usually only care about graphs upto their isomorphism class.

Deciding if two graphs are isomorphic is a pretty hard problem.

1.1 Some Common Graphs

- Complete graph, or clique, K_n: $V(K_n) = [n]$ and $E(K_n) = 2^{\binom{[n]}{2}}$.
- n-cycle C_n: $V = \mathbb{Z}_n$ and $E = \{\{i, i+1\} \mid i \in \mathbb{Z}_n\}$.
- n-path P_n: $V = \{0, 1, \ldots n\}$ and $E = \{\{i, i+1\} \mid i = 0, \ldots, n-1\}$.
- complete bipartite graph $K_{m,n}$: where $A = \{a_1, \ldots a_m\}$ and $B = \{b_1, \ldots, b_n\}$, $V = A \cup B$ and $E = A \times B$.
- n-cube Q_n: $V = \{0, 1\}^n$, E: $u \sim v$ if u and v differ in exactly one coordinate.

- The Petersen Graph:



The following is useful in proving things about the Petersen graph.

Problem 1.1. Show that for any vertex u and any other vertex v of the Petersen graph P, there is an *automorphism* f (that is, an isomorphism $f : P \to P$) such that $f(u) = v$. Graphs with this property are called *vertex transitive*.

1.2 Subgraphs and Complements

Given a graph G, another graph G' is a *subgraph* of G, written $G' \leq G$ if $V(G') \subset V(G)$ and $E(G') \subset E(G)$.

A subgraph G' of G is *spanning* if $V(G') = V(G)$, or *induced* if $E(G') = \{uv \in E(G) | u, v \in V(G')\}$. A subgraph G' of G is a *proper* subgraph of G if $E(G')$ is a proper subset of $E(G)$.

A subgraph of G isomorphic to K_n, C_n or P_n is called, respectively, an n-clique, n-cycle, or n-path of G.

Problem 1.2. Let G contain a cycle C and let there be a path of length at least k between vertices of C. Show that G contains a cycle of girth at least $\sqrt{k}$.

(This seems trivial when you first look at it, but the path can intersect the cycle, so there is something to do.)

A graph G is *connected* if for every two vertices u and v of G there is a $(u, v)-$path in G: a path in G with u and v as endpoints. A maximally connected subgraph of G is a *component*.

The *complement* $\overline{G}$ of a graph G is the graph defined by $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \overline{E(G)}$, (where the universe used for complementation is the set $\{uv \mid u \neq v \in V(G)\}$ of all possible edges on $V(G)$).

Problem 1.3. Show that a subgraph G' of G is induced if and only if $\overline{G'}$ is a subgraph of $\overline{G}$.

Problem 1.4. For a graph G show that either G or $\overline{G}$ is connected.

2 Degrees and Degree Sequences

The *degree* $\mathrm{d}(v)$ of a vertex of G is the number of edges it is in. The *graph score* of G is the multiset of the degrees of the vertices of G. We usually order the graph score of a graph so that its elements are in decreasing order, and then call it the *degree sequence* of the graph.

The minimum degree, $\delta(G)$, and the maximum degree, $\Delta(G)$, of G are the minimum and maximum degrees respectively in its degree sequence. If $\delta(G) = \Delta(G) = k$ then G is *k-regular*.

Here is a simple observation about degree sequence:

Theorem 2.1. *If $(d_1, d_2, \ldots, d_n)$ is the degree sequence of a graph, then the degree sum: $\sum_{i=1}^{n} d_i$ is equal to $2|E(G)|$.*

Corollary 2.2. *In any graph G, there is an even number of vertices of odd degree.*

A sequence $D = (d_1, d_2, \ldots, d_n)$ non-increasing integers is *graphic* if there is a graph G on the vertices $v_1, \ldots, v_n$ such that $d(v_i) = d_i$ for all $i = 1, \ldots, n$. Such a graph G is called a *realisation* of D. We saw above that one necessary condition for a sequence to be graphic is that it sums to an even number. But this is not sufficient: $(4, 2, 0)$ is not graphic.

Problem 2.1. Show that the following condition is also necessary for a sequence $(d_1, d_2, \ldots, d_n)$ to be graphic. For all $1 \leq k \leq n$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(k, d_i).$$

Note

It isn't so hard to prove. Look up a 2010 paper by Tripathi, Venugopalan and West for a nice short proof.

In fact, this condition is also sufficient. But this is fair bit harder to show:

Theorem 2.3. *(Erdős-Gallai '60) A sequence $(d_1, d_2, \ldots, d_n)$ is graphic if and only if $\sum_{i=1}^{k} d_i$ is even and for all $1 \leq k \leq n$,*

$$\sum_{i=1}^{n} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(k, d_i).$$

The Erdős-Gallai condition is what we use to decide if a sequence is graphic, but there is also a recursive method to decide it. It depends on the following idea:

Lemma 2.4. *(Havel-Hakimi) If a sequence $D = (d_1, d_2, \ldots, d_n)$ is graphic, then there is a graph with degree seqence D such that the maximum degree vertex is adjacent to the d_1 vertices of next highest degree.*

Problem 2.2. Prove this! And present it next class!

Thus to decide if a sequence D, such as $(3,3,2,2,2)$, is graphic, the *Havel-Hakimi algorithm* is as follows. While D is not all zeros,

- remove d_1, and
- remove 1 from the next d_1 degrees $d_2, \ldots, d_{d_1-1}$,

if possible.

The algorithm clearly terminates. If it terminates with an empty sequence, (or a sequence of zeros), then D is graphic. Otherwise, it is not graphic.

Starting with $(3,3,2,2,2)$, we get $(\not 3, 3-1, 2-1, 2-1, 2) = (2,2,1,1)$, and then $(1,0,1) = (1,1)$, and then $(0,0)$. So $(3,3,2,2,2)$ is graphic.

Problem 2.3. Decide, using Havel-Hakimi, whether or not the sequence $(5,5,5,3,2,2,1,1)$ is graphic.

Problem 2.4. Show that if loops are allow in a graph, a sequence $(d_1, d_2, \ldots, d_n)$ is graphic if and only if $\sum_{i=1}^{k} d_i$ is even. Show that if multiedges are allowed, but loops are not, then it is graphic if and only if $\sum_{i=1}^{k} d_i$ is even and $d_1 \leq \sum_{i=2}^{n} d_i$.

3 Graph Colouring

A *(proper) k-colouring* of a graph G is a function $f : V(G) \to [k]$ such that

$$u \sim v \Rightarrow f(u) \neq f(u).$$

If G has a k-colouring, it is *k-colourable*. The *chromatic number* $\chi(G)$ of G is the minimum k for which G is k-colourable. If $k = \chi(G)$, we also say that G is *k-chromatic*.

Problem 3.1. Show a graph is 2-colourable if and only if it has no odd cycles. Use this to give a polynomial time algorithm to decide if a graph is 2-colourable.

For any $k \geq 3$, it is an NP-complete problem to decide if a given graph is k-colourable. So do we give up?

Note

That a *decision problem* is NP-complete means that if one can verify a guessed solution in polynomial time, but (assuming that P $\neq$ NP) the best algorithm for solving the problem will take exponential time for infinitely many instances.
If we know a problem is NP-complete, we try to find

- restricted versions of it that are tractable (i.e., can be solved in polynomial time),
- polynomial time approximations or bounds,
- or polynomial time algorithms that solve most instances, or that solve the problem with high probability.

As deciding if a graph is 3-colourable is NP-complete, so hard. Determining the chromatic number of a graph is hard. So perhaps we try to show that for any graph the chromatic number is less than the largest degree, or show that planar graphs have low chromatic number, or give an polynomial algorithm that computes $\chi(G)$ within a factor of 2 at least 99 percent of the time.

There are several quick bounds we can get on the chromatic number.

The *clique number* $\omega(G)$ of a graph G is the maximum n such that K_n is a subgraph of G. The *independence number* $\alpha(G)$ is $\omega(\overline{G})$.

Note

The term *greedy* is usually used for algorithms that only try to be locally optimal, and that do not care about overall efficiency. They are often used to get loose bounds on a parameter.

Problem 3.2. Show that $\chi(G) \geq \omega(G)$ and that $\chi(G) \geq n/\alpha(G)$.

Problem 3.3. Show that $\chi(G) \leq \sqrt{2m + 1/4} + 1/2$.

It is easy to see that $\chi(G) \leq \Delta(G) + 1$, using the following greedy algorithm for a arbitrary ordering $v_1, \ldots, v_n$ of the vertices of G.

> **Greedy colouring:** For $i = 1 \ldots n$, colour v_i with the first colour not used on one of its already coloured neighbours.

This colouring requires at most d colours where $d-1$ is the maximum number of neighbours that any vertex has before it in the ordering. No vertex can have more than $\Delta(G)$ neighbours before it, so

$$\chi(G) \leq \Delta(G) + 1,$$

as needed.

Problem 3.4. Find a graph G and an ordering of its vertices such that the greedy colouring uses $\chi(G) + x$ colours. Show that there is an ordering of the vertices of any graph G such that the greedy colouring uses exactly $\chi(G)$ colours.

In showing $\chi(G) \leq \Delta(G) + 1$ we used this number d. For an ordering $v_1, \ldots, v_n$ of the vertices of G, the *degeneracy* d of the ordering is **one more than** the maximum, over all vertices, of the number of neighbours the vertex has before it in the ordering:

$$d = \max_{i=1}^{n} |N_G(v_i) \cap \{v_1, \ldots, v_{i-1}\}| + 1.$$

The *degeneracy* (or colouring number) $\mathrm{degen}(G)$ of G is the minimum of the degeneracy over all orderings of its vertices. The proof above actually showed

$$\chi(G) \leq \mathrm{degen}(G) \leq \Delta(G) + 1.$$

The degeneracy of a graph is easy to find. Consider the following ordering of the vertices of G.

Note

For a graph G and a subset S of the vertices of G, let the $G|_S$ be the induced subgraph of G on the vertex set S.

> **Reverse min-degree ordering:** For $i = n, \ldots, 1$ let v_i be a vertex of smallest degree in $G|_{v_n, \ldots, v_{i+1}}$.

Problem 3.5. a) Where d is the degeneracy of the reverse min ordering of a graph G, show that G has a subgraph H of min-degree $\delta(H) = d - 1$.

b) Show that if G has a subgraph H of min-degree $\delta(H) = d - 1$, then the degeneracy of any ordering of the vertices of G is at least d.

In answering this problem you have proved the following.

Proposition 3.1. *For any graph G, the degeneracy* $\mathrm{degen}(G)$ *is equal to*

- *the degeneracy of the reverse min-degree ordering of G, and*
- $\max\{\delta(H) \mid H \leq G\} + 1$.

As a corollary of the first part of this proposition we see that teh degeneracy can be found in polynomial time. As a corollary of the second part we get the following useful result.

Corollary 3.2. *Every graph G with $\chi(G) = k$ has a subgraph H with $\delta(H) = k - 1$.*

Problem 3.6. Prove this corollary in another way by showing that any k-critcial subgraph of G must have min-degree at least $k - 1$. A subgraph of G is k-critical if it is k-chromatic, but any proper subgraph of it is $k - 1$-colourable.

Here is one of those classic theorems of graph theory.

Theorem 3.3 (Brooks '41)**.** *Let G be connected. If G is not an odd cycle or a clique, then $\chi(G) \leq \Delta(G)$.*

Proof. The theorem is trivial if $\Delta(G) \leq 2$ so assume that $\Delta = \Delta(G) \geq 3$. We proceed by induction on $n = |G|$. Let v be vertex of G and let $H = G \setminus v$. I break the proof down into a series of claim, but am very brief on the details of the proofs of these. Please work out the details of the proofs of the claims.

Claim 3.4. *By induction H is Δ-colourable.*

Proof of claim. This is almost trivial. The only thing to observe is that if H contains a component that is an odd-cycle (and $\Delta = 3$) or a $(\Delta - 1)$-clique, then $\Delta(H) = \Delta - 1$. ◇

Fixing a Δ-colouring ϕ of H, our goal is to extend ϕ to v so that it is a Δ-colouring of G.

Problem 3.7. Prove the following claim. The phrase 'We may assume...' means, in this case, that if this is not true then we can easily prove our result. So assume that converse and show how to extend ϕ to v.

Claim 3.5. *We may assume that* $\mathrm{d}_H(v) = \Delta$ *and ϕ assigns a distinct colour to each vertex.*

Label the neighbours of v as $v_1, \ldots, v_\Delta$ and assume that $\phi(v_i) = i$ for all i. Let H_{ij} be the subgraph of H induced by vertices of colours i and j.

Claim 3.6. *For $i \neq j$, v_i and v_j lie in the same component C_{ij} of H_{ij}.*

Proof. Assume they do not, and let C be the component of H_{ij} that contains v_i. If we toggle the colours on C, that is, replace ϕ with the colouring ϕ' we get by setting

$$\phi'(x) = \begin{cases} i & \text{if } x \in C \text{ and } \phi(x) = j \\ j & \text{if } x \in C \text{ and } \phi(x) = i \\ \phi(x) & \text{if otherwise.} \end{cases}$$

Then ϕ' is a Δ-colouring and has $\phi'(v_i) = j = \phi'(v_j)$. So we are done by Claim 3.5. ◇

Problem 3.8. Show the following two claims.

Claim 3.7. *For $i \neq j$, C_{ij} is a path from v_i to v_j.*

Claim 3.8. *For distinct i, j, k, $C_{ij} \cap C_{jk} = v_j$.*

Now, if the vertices of $\{v_1, \ldots, v_\Delta\}$ are pairwise adjacent then G is a clique, and we are done, so wlog $v_1 \not\sim v_2$.

Let $u \sim v_1$ in $C_{1,2}$. Toggle the colours in C_{13} giving a new colouring ϕ' of H. We have that $\phi'(u) = \phi(u) = 2$, so $u \in C''_{23}$. But u is still in a $1, 2$-path from v_2, so is in C'_{12}, contradicting the last claim. □

4 Euler's Theorem and Hamilton's Theorem

Let x and y be vertices in a graph G. An xy-walk is a finite alternating sequence of vertices and edges:

$$x = x_0, e_1, x_1, e_2, e_2, \ldots, x_{\ell-1}, e_\ell, x_\ell = y$$

where $e_i = x_{i-1}x_i$ for $i = 1, \ldots \ell$. The length of the walk is ℓ, the number of edges. The walk is a *trail* if the edges are distinct, and is a *path* if the vertices are distinct, An xy-walk is *closed*, or a *circuit* if $x = y$.

Usually we just write the vertices

$$x_0 x_1 \ldots x_n,$$

or, if x and y are known, the edges

$$e_1 e_2 \ldots e_n.$$

Theorem 4.1. *Any xy-trail in a graph has an xy-path as a subgraph.*

Intuitively, we just start with an xy-trail an 'chop' out the repeated bits until it is a path. A lot of times it is very hard to write the intuitive proof rigorously. Try it.

Problem 4.1. Prove that any odd circuit in a graph has an odd cycle as a subgraph.

4.1 Girth and Distance

The *girth* of a graph is the minimum girth of all cycles in the graph.

The *distance*, $\mathrm{dist}(u, v)$ between two vertices u and v in a graph G is the length of the shortest uv-walk in G. Where $V = V(G)$, the *diameter* $\mathrm{diam}(G)$ of G is

$$\max_{u,v \in V} \mathrm{dist}(u, v),$$

and the *radius* rad(G) of G is

$$\min_{u \in V} \max_{v \in V} \operatorname{dist}(u, v).$$

A vertex u that minimises $\max_{v \in V} \operatorname{dist}(u, v)$ is called *central*.

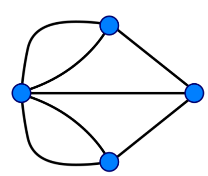
Problem 4.2. Show that for a graph G containing a cycle, the girth of G is at most $2\operatorname{diam}(G) + 1$.

Problem 4.3. How many vertices can a graph of radius k and maximum degree Δ have?

4.2 Eulerian Graphs

It is often told that the first problem of graph theory was the problem of the Bridges of Königsberg. In the fictional city of Königsberg (Problem: prove or disprove my unpopular claim that Königsberg is fictional.) there were seven bridges. The dandies of the city, on a slow Sunday, would walk about the town, playing a riddlish game. Starting whereever they pleased, they were to walk around, crossing every bridge, without crossing any bridge more then once. This game went on for centuries, and several liars claimed to have done it. Fermat once wrote in the margin of a book that it was simple enough. Until eventually, Euler proved it was impossible, and stabbed Fermat in a dual. (Prove that any of this is true.)

The city of Königsburg consisted of four dots, connected by seven bridges. It looked a lot like this:



How Euler proved it is lost in the depths of historical fiction. But our story inspires the following definitions.

An *euler trail* in a graph G is a trail containing all edges and vertices of G. If it is closed, it is an *euler circuit*. A graph is called *eulerian* if it contains an euler circuit.

The following is what Euler claims to have proved.

Theorem 4.2. *A graph is eulerian if and only if it is connected, and all of its vertices have even degree.*

Proof. That an eulerian graph is connected has all vertices of even degree is clear.

We show that it has an eulerian circuit. Indeed, as a length 1 walk uses no edges, there are trails in G. Let

$$W = v_0, e_1, v_1, e_2, \dots v_{n-1}, e_n, v_n$$

be the longest trail. We will show that W is an Eulerian circuit.

Claim 4.3. $v_n = v_0$

Proof. If not, the vertex v_n occurs only in an odd number of edges in W. As it has even degree, there is some edge $e_{n+1} = \{v_n, v_{n+1}\}$ that is not used in W. But then W can be extended by e_{n+1}, which contradicts the choice of W. ◇

Claim 4.4. *All vertices of G are in W.*

Proof. If some u is not in W, then as G is connected, there is a path P from u to some $v_i \in W$, none of whose edges are in W. But then the trail

$$u \xrightarrow{[} P]v_i, e_{i+1}, v_{i+1}, \dots e_n, v_0, e_1, v_1 \dots e_i, v_i$$

is a longer trail in G than W, contradicting the choice of W. ◇

Claim 4.5. *All edge of G are used in W.*

Proof. If some edge e is not used, then as W contains all vertices, $e = \{v_i, v_j\}$ for some v_i and v_j in W. We may assume that $i < j$. But then

$$v_i, e_{i+1}, \dots, v_j, e, v_i, e_i, v_{i-1}, \dots, v_0, e_n, v_n, \dots, e_{j+1}, v_j$$

is a longer trail in G than W. ◇

Thus W is an Eulerian tour of G. □

Observe that the same proof works if G is allowed to have loops or multiedges.

Problem 4.4. Show that every $4k$-regular graph has a decomposition into edge-disjoint paths of length 2.

4.3 Hamilton Paths and Cycles

This section follows West. (Probably other sections do too. I have to add such comments.)

A path, or cycle, in G is *hamilton* if it is bijective. A graph G is *hamiltonian* if it contains a hamiltonian cycle.

There were some nice conditions that characterised the eulerian graphs. We are not so lucky with hamilton graphs. In fact, the problem of deciding if a graph on n vertices is hamiltonian, is NP-complete. This doesn't mean we can't say anything about graphs that are hamiltonian. Indeed they must be connected, and have no leaves. We can say a bit more. We look at some sufficient conditions on a graph to imply that it is hamiltonian.

Problem 4.5. Find a hamilton path in Q_n.

Problem 4.6. Let G be a 3-regular graph that is uniquely 3-edge-colourable (there is a unique partition of the edges into 3 parts such that no part contains edges sharing a vertex). Show that G is hamiltonian.

Here is an early result of Dirac.

Theorem 4.6 (Dirac 1952)**.** *A graph G with n vertices is hamiltonian if $\delta(G) \geq n/2$.*

This is implied by the slightly stronger result of Ore.

Theorem 4.7 (Ore 1960)**.** *A graph G with $n \geq 3$ vertices is hamiltonian if for every two vertices $x \neq y$ with $x \not\sim y$, we have $\mathrm{d}(x) + \mathrm{d}(y) \geq n$.*

Proof. Towards contradiction, let G be a maximal counterexample on $n \geq 3$ vertices. So G is not hamiltonian, and satisfies

$$(*) \qquad x \not\sim y \Rightarrow \mathrm{d}(x) + \mathrm{d}(y) \geq n,$$

but $G + xy$ is hamiltonian for any $x \not\sim y$. (We may assume that G is not complete, as complete graphs are clearly hamiltonian.) Let $H = G + xy$ for some $x \not\sim y$ in G. Then H is hamiltonian, and any Hamilton cycle contains xy. So there is a hamilton path

$$x = x_1 \sim x_2 \sim \dots x_n = y.$$

Now if $x_1 \sim x_i$ and $x_{i-1} \sim x_n$, then

$$x_1 \sim x_2 \sim \cdots \sim x_{i-1} \sim x_n \sim x_{n-1} \sim \cdots \sim x_i \sim x_1$$

is a hamilton cycle not using xy so in G, which is a contraction. So for each of xs $\mathrm{d}(x)$ neighbours x_i in $\{x_2, \dots, x_{n-1}\}$ (recall that $x \not\sim y = x_n$), x_{i-1} is not a neighbour of y. So

$$\mathrm{d}(x) + \mathrm{d}(y) = \mathrm{d}(x_1) + \mathrm{d}(x_n) \leq \mathrm{d}(x_1) + (n - 1 - \mathrm{d}(x_1)) \leq n - 1$$

This contradicts $(*)$, so the theorem is true. □

Both Dirac's and Ore's theorem give us conditions on the degree sequence of a graph that are sufficient to imply the existence of a Hamilton cycle. Are there others? There are, we look at one more generalisation, which is best possible in the sense that for any weakening of the condition (strictly in terms of the degrees) there is a non-hamiltonian graph satisfying the weakening.

First, Bondy and Chvatal both observed that the proof of the Ore condition actually proves the following.

Lemma 4.8. *Let G be a simple graph on n vertices, and x and y be non-adjacent vertices satisfying the (BC) condition*

$$deg(x) + deg(y) \geq n.$$

Then G is hamiltonian if $G + xy$ is.

This leads to the following definition.

Definition 4.9. For a graph G, the (hamilton) closure $c(G)$ is the graph constructed by recursively adding the edge xy for non-adjacent vertices satisfying (BC).

To see that this definition is well defined, assume that $G_1 = G + e_1 + \cdots + e_a$ and $G_2 = G + f_1 + \cdots + f_b$ are two closures of G under the definition. We show that all of the e_i are among the f_i. If not, there is a first e_i not in G_2. Then

$$G_1^* = (G + e_1 + \cdots + e_{i-1}) \leq G_2$$

so the degree sum of the endpoints of e_i in G_2 is greater than that in G_1^*, and so G_2 is not closed, a contradiciton.

It follows from the lemma that a graph is hamilton if and only if its closure it. This is a nice condition, but doesn't settle the problem of deciding whether or not a graph is hamiltonian. It is still difficult to decide if the closure is hamiltonian. However, if we are lucky, the closure will be a clique, and then we know it is hamiltonian. Using this idea, Chvatal showed the following.

Theorem 4.10 (Chvatal '72)**.** *Let G be a graph on $n \geq 3$ vertices, with degree sequence $D = (d_1, \ldots, d_n)$ such that for all $i < n/2$ either $d_{n-i+1} > i$ or $d_{i+1} \geq n - i$. Then G is hamiltonian.*

Proof. Let G be a graph satisfying this condition, we show that $c(G)$ is a clique. As cliques are hamiltonian, this is enough.

Towards contradiction assume that $c(G)$ is not a clique, so there are non adjacent vertices u and v. Where $d(v)$ denotes the degree of v in $c(G)$, let u and v be non-adjacent vertices that maximise $d(u) + d(v)$. Assume, wlog, that $m := d(u) \leq d(v)$. As $d(u) + d(v) < n$ because $u \not\sim v$, we have $m \leq n/2$ and $d(v) < n - m$. We show that $d_{n-m+1} \leq m$ and $d_{m+1} < n - m$ to contradict the assumption.

First we show that $d_{n-m+1} \leq m$. Indeed, as $d(v) < n - m$, v has at least m non-neighbours, and by choice of u and v, all have degree at most $d(u) = m$. So $d_{n-m+1} \leq m$.

Now we show that $d_{m+1} < n - m$. Indeed, as $d(u) = m$, u has exactly $n - m - 1$ non-neighbours, which have degree at most $d(v) < n - m$. And u itself has degree at most this, so there are at least $n - m$ vertices with degree at most $n - m$. Thus $n - m > d_{n-(n-m)+1} = d_{m+1}$, as needed. □

Problem 4.7. Show that every connected graph G with $n > 2\delta(G)$ vertices has a path of length $2\delta(G)$. Deduce that every $2k$-regular graph on $4k + 1$ vertices is hamiltonian.

5 Trees

A graph is *acyclic* or a *forest* if it has no cycles. A *tree* is a connected acyclic graph. A *leaf* of a tree is a vertex of degree one.

A *cut edge* of a graph G is any edge whose removal increases the number of components of G.

Lemma 5.1. *An edge in a graph G is a cut edge if an only if it is not in any cycles in G.*

Theorem 5.2. *A graph T is a tree if and only if it is connected and every edge is a cut edge.*

Problem 5.1. Prove Lemma 5.1, and use it to prove Theorem 5.2.

A *spanning tree* of a graph G is a spanning subgraph which is a tree.

Theorem 5.3. *A graph G is connected if and only if it has a spanning tree.*

Proof. Let G be a graph with a spanning tree T. Let x and y be any two vertices in G. As T is spanning, they are also in T. As T is connected, there is an xy-path $P = e_1 e_2 \dots e_d$ in T. But $E(T) \subset E(G)$ so P is in G as well, that is, it is an xy-path in G. This was for any $x, y \in V(G)$, so G is connected.

On the other hand assume that G is connected. Let H be a minimal connected spanning subgraph of G, that is, a connected spanning subgraph, which itself has no connected spanning proper subgraph. By minimality, H has no cut edges, so is a tree by Theorem 5.2. □

Lemma 5.4. *Any tree has a leaf.*

Lemma 5.5. *For any tree T and any leaf v of T, the graph $T' = T \setminus \{v\}$ we get from T by removing v (and any incident edges) is also a tree.*

Theorem 5.6. *For a tree T, $|V(T)| = |E(T)| + 1$.*

Problem 5.2. Prove Lemmas 5.4 and 5.5 and use them to prove Theorem 5.6.

Theorem 5.7. *Let T be a graph with n vertices and m edges. Then the following are equivalent.*

i. T is a tree; ie. it is connected and acyclic.

ii. T has no cycles, and $m = n - 1$.

iii. T is connected, and $m = n - 1$.

iv. T is connected, but every edge is a cut edge.

v. *Any two vertices of G are connected by a unique path.*

Proof. From Theorem 5.2 we have the equivalence of (i) and (iv). By Theorem 5.6 we have that (i) implies (ii) and (iii). We show that (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv). This shows the equivalence of the first four statements. As an exercise, show that statement (v) is also equivalent.

(ii) $\Rightarrow$ (i): Assume that G has no cycles, so each component of G is a tree. For each component C we thus have that $|V(C)| = |E(C)| + 1$. So $n = m + c$ where c is the number of components. By $n = m + 1$, there is only one component, so it is a tree.

(ii) $\Rightarrow$ (iv): Assume that G is connected and has $n = m + 1$. If there is some edge e that is not a cut-edge, then $G' = G \backslash \{e\}$, the graph we get by removing e, is connected, and so contains a spanning tree T. Thus $n = |V(G')| = |V(T)| = |E(T)| + 1 \leq |E(G')| + 1 = m + 2$, but this is a contradiction.

□

5.1 Cayley's Formula

How many different trees are there on n vertices?

Theorem 5.8 (Cayley's formula)**.** *There are $T_n = n^{n-2}$ different labelled trees on n vertices.*

By labelled, we mean that the path $1 \sim 2 \sim 3$ on the vertex set $\{1, 2, 3\}$ is different from the path $2 \sim 1 \sim 3$. (But it is the same as the path $3 \sim 2 \sim 1$.) So as a corollary, we get that there are between $n^{n-2}/n!$ and n^{n-2} non-isomorphic trees on n vertices. We can count them using generating functions, we might get to that later.

In this section we give two proofs of Cayleys theorem. The first, due to Prüffer, we only sketch.

5.1.1 Prüffer's Proof

This is a proof by bijection. Observe that the number of $(n-2)$-element sequences $s \in [n]^{n-2}$ of integers from the set $[n] = \{1, 2, \ldots, n\}$ is exactly n^{n-2}. We show a correspondence between these sequences and the labelled trees on n vertices.

Prüffer Encoding

Given a tree T_1 with vertices labelled with the integers $[n]$, construct $s \in [n]^{n-2}$ as follows.

For $i = 1, \ldots, n-2$ do

- Let ℓ_i be the lowest labelled leaf in T_i.
- Let s_i be the unique neighbour of ℓ_i in T_i.
- Let $T_{i+1} = T_i \setminus \ell_i$.

Prüffer Decoding

Given an sequence $s = (s_1, \ldots, s_{n-2})$, construct a tree T on the vertex set $[n]$ as follows. Let L_1 be empty, and let $Z_i = (s_i, s_{i+1}, \ldots s_{n-2})$ for $i = 1, \ldots, n-2$.

For i from 1 to $n-2$ do

- Let ℓ_i be lowest number not in $L_i \cup Z_i$.
- Add the edge $\ell_i s_i$.
- Let $L_{i+1} = L_i \cup \{\ell_i\}$.

When the above algortihm terminates, there are two vertices not in L_{n+1}. Add the edge between them.

Problem 5.3. Show that Prüffer encoding and Prüffer decoding are inverse operations.

5.1.2 Pitman's Proof

This beautiful little double counting proof of Cayley's Formula by Pitman, can be found in 'Proofs from the Book' by Aigner and Zeigler. It's a nice book.

We need some definitions.

An *oriented graph* $\vec{G}$ is a graph G in which we apply a orientation, or direction, to each edge. An oriented edge is often referred to as an *arc.* We write $u \to v$ to denote that there is an arc $a = (u, v)$ from u to v. We say that u is the *source* or *start* of the arc, and v is the *terminus* or *end* of the arc. We also say a is an *in-arc* of v and an *out-arc* of u. The *in-degree* of a vertex in an oriented graph is the number of in-arcs, and the *out-degree* is the number of out-arcs. A path $P = v_1 \sim v_2 \sim \cdots \sim v_d$ is a *directed* path, directed from v_1 to v_d if it is given the following orientation: $v_1 \to v_2 \to \cdots \to v_d$. An oriented cycle is directed if every vertex has in-degree one and out-degree one.

A *rooted tree* is a tree in which there is a particular vertex designated as the *root.*

We also need some easy observations. Given a tree T, choosing a root r defines a unique orientation on the tree in which for every vertex $u \neq r$ the unique path between u and r in T is a directed path from u to r. As such, we call an oriented tree a *rooted tree* if there is some vertex r with such that every arc is directed towards r.

Problem 5.4. Show that an oriented tree is a rooted tree with root r, if and only if every vertex but r has out-degree one, if and only if it has max out-degree 1.

Pitman's proof of Cayley's theorem comes from counting the possible constructions of a rooted tree in two ways.

Let F_n be the number of sequences $(a_1, a_2, \dots, a_{n-1})$ of arcs on the set of vertices $[n]$, such that the oriented graph $\vec{G}$ with vertices $[n]$ and arcs $a_1, a_2, \dots, a_{n-1}$ is a rooted tree.

First we observe that $F_n = T_n \cdot n!$. Indeed, we choose a tree in T_n ways, and a root in n ways. The $n-1$ arcs are determined, but the order they come in in the sequence can be now chosen in $(n-1)!$ ways. Together, this is $T_n \cdot n!$, as needed.

On the other hand we can construct a sequence as follows. Let a_1 be any arc. This can be chosen in $n(n-1)$ ways. At step i, let a_i be any arc such that the oriented graph with arcs $a_1, \dots, a_i$ has no vertex of out-degree greater than one, and no cycles.

Observe that the arc we add at step i must go between components, or it makes a cycle. So it reduces the number of components by one. Thus before adding a_i there are $n-i$ components. Further, as each component has max out-degree 1 and is a tree, it has a unique vertex, its root, with out-degree 0. The arc a_i may end at any vertex v, and must then begin at the root of any component other than the component containing v. So can be chosen in $n(i-1)$ ways. This gives

$$F_n = \prod_{i=1}^{n-1} n \cdot (n-i) = n^{n-1} \cdot (n-1)! = n^{n-2} \cdot n!.$$

So $T_n \cdot n! = F_n = n^{n-2} \cdot n!$, which yields Cayley's formula: $T_n = n^{n-2}$.

5.2 The Matrix Tree Theorem

We have counted the number of trees on $[n]$ vertices. We can view each as a spanning subtree of K_n. This was hard enough for K_n. It should be harder for a general graph G on n vertices. And it is. But it turns out, that the number of spanning subtrees of G is counted by the determinant of a particular $m \times m$ matrix which we get by multiplying two $n \times m$ matrices which we use to encode our graph. (Recall n is the number of vertices and m is the number of edges.) Multiplying the matrices takes $O(m^2 n) = O(n^5)$ operations and computing the determinant takes $O(n^4)$, so this is not a bad calculation. This is the subject of this subsection. We start by looking at a couple matrices associated to a graph.

Recall that $[x]_{m,n}$ is the $m \times n$ matrix with entry $x_{i,j}$ in then i^{th} row and j^{th} column.

So, for example, when

$$M = [m]_{3,4} = \begin{bmatrix} 3 & 4 & 5 & 2 \\ 1 & 0 & 3 & 3 \\ 0 & 2 & 1 & 7 \end{bmatrix}$$

then $m_{1,3} = 5$.

If $L = [x]_{\ell,m}$ and $M = [y]_{m,n}$ then the product $LM = [z]_{\ell,n}$ is matrix defined by

$$z_{i,j} = \sum_{\alpha=1}^{m} x_{i,\alpha} y_{\alpha,j}.$$

It will be convenient to index the rows and columns of a matrix M by sets R and C rather than by the set $[m]$ and $[n]$. In this case we write $M = [x]_{R,C}$ to mean that $x_{r,c}$ is the entry in row r and column c . For a subset $R' \subset R$ let $M^{R'}$ be the submatrix of M consisting of rows in R' and for We write M^r for the row vector $M^{\{r\}}$ and M_c for the column vector $M_{\{c\}}$. So the product $LM = [z]_{R,C}$ can be defined succincty by

$$z_{r,c} = L^r \cdot M_c.$$

5.2.1 The Adjacency Matrix $A = A(G)$

The *adjacency matrix* $A = A(G)$ of a graph G with vertex set V is the is the $n \times n$ matrix $A = [a]_{V,V}$ where

$$a_{i,j} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{if } i \not\sim j. \end{cases}$$

Observe that this is a symmetric matrix. The number of '1's in the v^{th} row A^v or column A_v is the degree of vertex r.

Problem 5.5. Where $A = A(G)$, show that the (u, v) entry $(A^2)^u_v$ of $A^2 = AA$ is the number uv-walks of length 2 in G, and in particular that the (u, u) entry of A^2 is the degree of i in G. Show, more generally that the (u, v) entry of A^k is the number of uv-walks of length k in G.

5.2.2 The Incidence Matrix $M = M(G)$

The *incidence matrix* $M = M(G)$ is the $m \times n$ matrix $M = [m]_{E,V}$ such that

$$m_{e,v} = \begin{cases} 1 & \text{if } v \in e \\ 0 & \text{if } v \notin e. \end{cases}$$

Notice that where $\mathbf{1}$ is the appropriately sized vector of ones, $M\mathbf{1} = 2\mathbf{1}$. For all our applications, it will be more useful to assume an arbitrary orientation on G, and to define $M = M(G) = M(\vec{G})$ to reflect that orientation:

$$m_{e,v} = \begin{cases} -1 & \text{if } v \text{ is the start of } e \\ 1 & \text{if } v \text{ is the terminus of } e \\ 0 & \text{otherwise} \end{cases}$$

As we are not allowing loops, this is well defined, and is skew-symmetric. Moreover, we now have that

$$M\mathbf{1} = 0.$$

Infact, we have the following more general fact.

Problem 5.6. Show that $M_s\mathbf{1} = 0$ if and only if the subgraph G graph induced by S is a union of components of G. (That is, there are no edges between S and $V \setminus S$.)

Now consider the matrix equation $M\mathbf{x} = \mathbf{b}$.

The column vector $\mathbf{b}$ assigns a value b^e to each arc e of G. A solution $\mathbf{x}$ assigns a value x_v to each vertex v. Where $e = (u, v)$ the e^{th} row of this equation becomes $M^e\mathbf{x} = b^e$ which expands to

$$-x_u + x_v = b^e = b^{(u,v)}.$$

Rearranging this we get that $x_v = x_u + b^{(u,v)}$.

Problem 5.7. If T is a subtree of G then $M^{E(T)}_{V(T)}\mathbf{x} = b$ has a solution for any b.

Problem 5.8. If C is the ℓ-cycle $v_1e_1v_2e_2\dots v_\ell e_\ell v_1$ in G then $M^{E(C)}_{V(C)}\mathbf{x} = b$ if and only if $\sum_{i=1}^{\ell} b^{e_i}p_i = 0$ where p_i is 1 or -1 depending on whether e_i is a forward arc (v_i, v_{i+1}) or a backwards arc (v_{i+1}, v_i).

Fix an arbitrary vertex v of G and let $M_* = M(G)_{V(G)\setminus\{v\}}$.

Lemma 5.9. *For a set S of $n-1$ edges of G, M^S_* has full rank if and only if S is a spanning tree of G.*

Proof. For a subset S of $n-1$ edges of G, M^S is an $(n-1)\times n$ matrix, so has rank at most $n-1$.

If S is a spanning tree, then we have by Problem 5.7 that M^S has rank exactly $n-1$. As $M^S\mathbf{1} = \mathbf{0}$, we have that the column M^S_v is a linear combination of the other columns, and so M^S_* also has $n-1$ independent columns. Thus the the $(n-1)\times(n-1)$ matrix M^S_* has full rank.

On the other hand, if S it is not a spanning tree, then it has a cycle, and so by Problem 5.8, M^S has rank less than $n-1$. Thus M^S_* also has rank less than $n-1$. □

Problem 5.9. Show that for a set S of $n-1$ edges of G, $\det(M_*^S) = \pm 1$ if S is a spanning tree of G, and is 0 otherwise.

5.2.3 The Matrix Tree Theorem

Recall the Binet Cauchy formula for an $m \times n$ matrix A and an $n \times m$ matrix B when $n \leq m$:

$$\det(AB) = \sum_{S \subset [m], |S| = n} \det(A_S B^S)$$

Recall that the *transpose* M^T of an $m \times n$ matrix $M = [m]_{R,C}$ is the $n \times m$ matrix $[m]_{C,R}$ we get by interchanging rows and columns.

With the Binet Cauchy formula we can prove the following.

Theorem 5.10. *Given a connected graph G and a vertex $v \in V(G)$, let $M = M(\vec{G})$ for some orientation $\vec{G}$ of G and let $M_* = M_{V(G)\setminus\{v\}}$. Then the number of spanning trees of G is* $\det(M_*^T M_*)$.

Proof.

$$\begin{aligned}
\det(M_*^T M_*) &= \sum_{S \subset E, |S| = n-1} \det((M_*^T)_S M_*^S \\
&= \sum_{S \subset E, |S| = n-1} \det(M_*^S)^2 \\
&= \sum_{S \subset E, |S| = n-1} \begin{pmatrix} (\pm 1)^2 & \text{if } S \text{ is a tree} \\ 0^2 & \text{otherwise} \end{pmatrix} \\
&= \text{the number of spanning trees of } G.
\end{aligned}$$

□

Problem 5.10. The *Laplacian matrix* of a graph G is the $n \times n$ matrix $L(G) = MM^T$ where $M = M(\mathbf{G})$ for some orientation $\mathbf{G}$ of G. Show that $L(G) = [x]_{V,V}$ satisfies

$$x_{u,v} = \begin{cases} \mathrm{d}(u) & \text{if } u = v \\ -1 & \text{if } u \sim v \\ 0 & \text{if neitherofthose,} \end{cases}$$

independent of the ordering of G.

Problem 5.11. Show how Cayley's theorem follows from the Matrix Tree Theorem.

6 Connectivity

A graph $G \neq K_k$ is k-connected if for any set S of fewer than k vertices, $G \setminus S$ induces a connected graph. The *connectivity* $\kappa(G)$ of a graph G is the maximum k for which it is k-connected.

Note

What is the maximum connectivity of a graph of min-degree δ? Right: $\delta - 1$. We can disconnect the graph by removing all the neighbours of a vertex of minimum degree.

What is this weird little condition '$G \neq K_k$' in the definition of k-connected? The point is that we do not want K_k to be k-connected. When we defined a graph, we insisted that it had a non-empty set of vertices, so really if we take $S = V(G)$ then $G \setminus S$ does not induce a connected graph. So we do not really have to state $G \neq K_k$. But in some books, the *null graph*, a graph on no vertices, is taken to be connected. So we add this condition to avoid misunderstanding.

Why do we not want K_k to be k-connected? It acts more like a graph of connectivity $k-1$. We don't want K_2 to be 2-connected while all other trees have connectivity 1. If K_k was k-connected, then the above question about min-degree would not hold. Menger's theorem, which we show below, would not hold. Lots of things would become messy.

6.1 2-connected graphs

In this section we give a useful characterisation of 2-connected graphs.

For distinct vertices a,b and c in a connected graph G, a *c avoiding ab-path* is an ab-path that does not contain c. The vertex c is an *ab-separator* if there are no c-avoiding ab-paths.

Problem 6.1. Prove the following.

i. A connected graph G is 2-connected if and only if there is no ab-separator for any two vertices a and b.

ii. If a and b are in a cycle in G then there is no ab-seperator.

An *ear* of a graph H is a path $P = p_0 \sim \cdots \sim p_d$ with $V(H) \cap V(P) = \{p_0, p_d\}$.

Theorem 6.1. *A graph G is 2-connected if and only if it is a cycle or can be constructed from a proper 2-connected subgraph H by adding an ear.*

Proof. Let G be such a graph. If G is a cycle we are done. So we assume that it can be constructed from a 2-connected subgraph H by adding an ear P.

Problem 6.2. Show that G is 2-connected

On the other hand, let G be 2-connected. Then G has a cycle, and so has some maximal subgraph H constructed as above. By its maximality, H must be an induced subgraph. So, unless we are done, there is some vertex $a \in G \setminus H$ with an edge ab to H. Since G is 2-connected there isa b-avoiding ah-path P for some vertex h of H (take a minimal one). But then $b \sim a \overset{[}{\longrightarrow} P]h$ is an ear of H, and its union with H is a larger graph constructible as above. □

Problem 6.3. Using Theorem 6.1 prove the converse of Problem 6.1(ii).

We generalise this now.

6.2 Menger's Theorem

For a graph G and an edge $e = xy$ of G, the graph G/e which we get from G by *contracting* e is the graph with vertex set

$$V(G) \setminus \{x, y\} \cup \{v_e\}$$

and edgeset

$$\{uv \in E(G) \mid u, v \in V(G)\} \cup \{uv_e \mid ux \in E(G)\} \cup \{uv_e \mid uy \in E(G)\}.$$

For sets A and B of G a path with an endpoint in each of A and B is an AB-path. (If A and B share a vertex v then the length 0 path v is an AB-path.) An AB-path is X-avoiding for a set X if it contains no vertices of X. A set X is an *AB-separator* if there are no X-avoiding AB-paths.

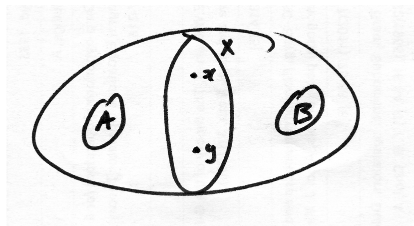
Clearly if two sets A and B in $V(G)$ have a AB separator X with $|X| = k$ there can be at most k disjoint AB-paths. Menger showed the following.

Theorem 6.2 (Menger 1927)**.** *For vertex sets $A, B \subset V(G)$ of a graph G if the smallest AB-separator has has k vertices, then G contains k disjoint AB-paths.*

Proof. The proof is an induction on the number of edges. With no edges the result is trivial. So assume G has an edge $e = xy$ and the result holds for $G' = G/e$.

Let A and B be sets in G. Let $A' = A \cap V(G')$ and add v_e to it if either of x or y are in A. Construct B' from B similarily. If G' has k disjoint $A'B'$-paths then G clearly has k disjoint AB-paths, so we may assume that G' has at most $k-1$ disjoint $A'B'$-paths. By the induction hypothesis, G' contains an $A'B'$-separator X' with $k-1$ vertices. If v_e is not in X' then it is an AB-separator in G, which is impossible, so v_e is in X'. Let $X = X' \setminus \{v_e\} \cup \{x, y\}$. This is an AB-separator in G having k vertices.

Now, consider the graph $H = G \setminus e$.



Any AX-separator in H is an AX-separator in G, so is an AB-separator in G, so has at least k vertices. Similarily and XB-separator in H has at least k vertices. By induction there are k disjoint AX-paths in H and k disjoint XB paths. As $|X| = k$ we can patch them together at X, and get k disjoint AB-paths. $\square$

Two ab-paths P and Q in a graph are *internally disjoint* if $V(P) \cap V(Q) = \{a, b\}$.

Problem 6.4. Show that a graph (with at least $k+1$ vertices) is k-connected if and only if for every pair (a, b) of distinct vertices it contain a set of k ab-paths, any two of which are internally disjoint.

A graph is *k-edge-connected* if for any set S of $k-1$ edges, $G \setminus S$ is connected.

Problem 6.5. Show that a graph G is k-edge-connected if and only if it contains k edge-disjoint paths between any two vertices.

Hint Consider the *line graph* $L = L(G)$ of G. It is defined by $V(L) = E(G)$ and $ef \in E(L)$ if $|e \cap f| = 1$.

7 Planar Graphs

A graph G is *planar* if it can be drawn in the plane with no edges crossing. A drawing of the graph in the plane with no edges crossing is called a *plane graph* or an *(planar) embedding of* G.

Problem 7.1. Show that K_4 is planar but that K_5 is not.

We can show that K_5 is not planar by ad-hoc arguments, but such arguments difficult for larger graphs.

7.1 Euler's Formula and applications

A planar embedding of a graph G defines *faces* in the plane: the little bits of plane you get by cutting along all the edges of the embedded graph. Where

$|V(G)| = n$ and $|E(G)| = m$ and f is the number of faces. Euler showed the following fundamental identity.

Theorem 7.1 (Euler's Formula)**.** *For any embedding of a connected graph G in the plane,*

$$n - m + f = 2.$$

Proof. The proof is by induction on m. As G is connected, $m \geq n - 1$. If $m = n - 1$ then G is a tree, so there is only one face, and the identity holds. This is our base case.

Now assume that $m \geq n$. Then there is an non-cut edge e. Remove it. Its removal joins two faces, so reduces f by one, but also reduces e by one. So the identity holds from the induction hypothesis. □

Corollary 7.2. *If G is connected and planar and $n \geq 3$, then*

$$m \leq 3n - 6.$$

Proof. Count the number N of pairs (e, r) where e is an edge, and r is a face with e on the boundary. We show that

$$f \leq \frac{2}{3}m,$$

plugging it into $2 = n - m + f$ gives

$$2 \leq n - m + \frac{2}{3}m = n - m/3,$$

so $m \leq 3n - 6$ as needed.

If there is only one face, then $f \leq \frac{2}{3}m$ hold by the assumption that G is connected and $n \geq 3.$, So we may assume that there are at least 2 faces. We count the number N of pairs (F, e) where F is a face containing the edge e. As every face has at least 3 edges, and every edge is in at most 2 faces, we have

$$3f \leq N \leq 2m$$

which gives us $f \leq \frac{2}{3}$. □

Problem 7.2. Use this corollary to show that K_5 is not planar. Improve it to show that $K_{3,3}$ is not planar.

Problem 7.3. Show that any planar graph has a vertex of degree at most 5.

7.1.1 Dual Graphs

Given an embedding of a graph G we construct a (multi-)graph G^*, called the *dual of* G, from G as follows.

The vertices of G^* are

$$V(G^*) = \{v_f \mid f \text{ is a face of } G\ \}.$$

The edges are defined as follows. For every edge e of G let $e^* = v_f v_g$, where f and g are the faces of G incident to e. Then $E(G^*)$ is the **multiset**

$$E(G^*) = \{e^* \mid e \in E(G)\}.$$

(Note that both $V(G^*)$ and $E(G^*)$ depend on the particular embedding we chose for G.) Not only is G^* a multigraph, it can have loops.

Problem 7.4. Find conditions on a graph G so that its dual G^* has no loops. Find conditions on G so that its dual, under any embedding, has no multi-edges.

Problem 7.5. Show that the dual of an embedded graph is planar.

Problem 7.6. Show for a connected embedded planar graph G, the dual of the dual is G. Show this isn't true for a non-connected graph.

Theorem 7.3. *Let G be planar and G^* the dual of an embedding of G. Then G is bipartite if and only if G^* is eulerian.*

Proof. (Sketch) One way is easy. Let G be a bipartite graph embedded in the plane. We show that every vertex of G^* has even degree. We may assume that G has no leaves, as the remove of a leave changes the degree of a vertex of G^* by 2. For any vertex v_f of G^*, the walk around the boundary of f is a closed walk. As G is bipartite, it is even, so v_f has even degree. This was for any vertex of G^*, so G^* is euleriain.

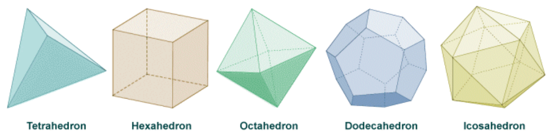
On the other hand, assume that G^* is eulerian. We want to show that every cycle of G is even. (This is clear of the boundary of faces under a given embedding but not of every cycle.) Fix an embedding of G.

Let C be a cycle in G, and let $A \subset F(G)$ be the set of faces inside C relative to the embedding. Then C is the symmetric difference of the edge sets of faces in A. Each face has a even number of edges, so so does their symmetric difference. Thus C is even. □

7.1.2 The Platonic Solids

Recall that a *platonic solid* is a 3-dimensional convex solid bounded by f copies of the same flat face, such that every vertex (point of intersection of more than two faces) intersects the same number of faces.

There are 5 platonic solids:



It isn't the original proof, but Euler's formula gives a nice proof that these are the only platonic solids.

The set of vertices of such a solid and the set of edges (intersections of 2 faces) make up a regular graph, and it is not to hard that this graph can be drawn in the plane so that every face has the same number of bounding edges.

(Recall a graph is regular if every vertex has the same degree.)

Theorem 7.4. *The only connected regular graphs that can be draw in the plane with each face having the same number of edges, are the* 5 *we get from the platonic solids above, and cycles.*

Proof. Let G be a planar graph that is regular with degree d, and assume there is a planar drawing such that every face has b edges. Let f be the number of faces.

We have seen that the sum of the degrees of a graph is $2m$, so

$$2m = dn.$$

Summing the number of edges per face over all the faces, we get fb and clearly every edges was counted twice, so

$$fb = 2m.$$

By Euler's formula we then get that

$$2 = n - m + f = \frac{2m}{d} - m + \frac{2m}{b}$$

yielding

$$\frac{1}{m} = \frac{1}{d} - \frac{1}{2} + \frac{1}{b}$$

and so

$$\frac{1}{d} + \frac{1}{b} = \frac{1}{2} + \frac{1}{m}.$$

If $b, d \geq 4$ then $\frac{1}{d} + \frac{1}{b} \leq \frac{1}{2}$ so there is no solution to above equation. If one of b or d is 3 and the other is at least 6, the again there is no solution. (b can never be less than 3 as there is no face with 2 edges; if d is less than 3 then G can only be a cycle.) If $b = 3$ then the d can be 3 (tetrahedron), 4 (octahedron) or 5 (icosohedron). If $d = 3$ then b can be 3 (tetrahedron), 4 (cube), or 5 (dodecahedron). □

Problem 7.7. Let G be a plane graph in which every edge is coloured red or blue. For each vertex v, cyclically order the edges incident to v by how you encounter them as you travel in a circle clockwise around the vertex. (This is a cyclic ordering, so $(r, b, b, r, b) = (b, b, r, b, r)$.) Show that there is some vertex v in which all incident red edges occur consecutively. (If a vertex has 0 or 1 red edge, then they are necessarily consecutive.)

7.2 Kuratowski's Theorem

So K_5 and $K_{3,3}$ are not planar. Is the Petersen graph? That K_5 is not planar tells us that a lot of other graphs are not planar.

7.2.1 Graph Minors

For a graph G, any graph M that can be formed from G by a series of vertex deletions, edge deletions, and edge contractions, is a minor of M.

It should be pretty clear that a graph with a K_5 or $K_{3,3}$ minor is not planar. Can you think of any other graphs that are not planar? Please don't look ahead. It will ruin the surprise.

Okay. Here's the surprise.

Theorem 7.5 (Kuratowski's Theorem)**.** *A graph is planar if and only if it contains no K_5 or $K_{3,3}$ minor.*

Proof. Assume G has no K_5 or $K_{3,3}$ minor.

The proof is by induction on $n + m$. The base case is trivial.

Claim 7.6. *We may assume G is connected.*

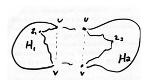
Proof. Clearly G is planar if and only if each component is, and is K_5 and $K_{3,3}$-free if and only if each component is. ◇

Claim 7.7. *We may assume that G is 2-connected*

Proof. If G has a cut vertex x, we separate G at x getting a set of proper subgraphs of G each containing x. As G was $K_5, K_{3,3}$ free so are thse subgraphs, so we can embed them by induction. Further, we cnan embedd them with x on the outer face. Moving the copies of x together, thi sgives us an embedding of G. ◇

Claim 7.8. *We may assume that G is 3-connected.*

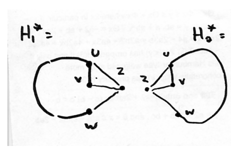
Proof. If not there is a seperator $\{u, v\}$ seperating H_1 and H_2 with $H_1 \cup H_2 = G$ and $H_1 \cap H_2 = \{u, v\}$. If $u \sim v$ then we can embed H_1 and H_2 and glue the embeddings together on $u \sim v$. So assume that $u \not\sim v$. Let $z_i \in H_1 \setminus \{u, v\}$. By Menger, there are two internally disjoint paths from z_1 to z_2; one must go through u and the other through v. These paths in H_2 imply that $H_1 + \{u, v\}$ is a minor of G and in H_1 imply that $H_2 + \{u, v\}$ is a minor. They are therefore both free of K_5 and $K_{3,3}$ minors, so can be embedded and then joined on $u \sim v$.



◇

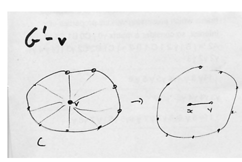
Claim 7.9. *For all edges uv, $G' := G \setminus uv$ is 3-connected.*

Proof. If not, there is a seperator $\{v_{uv}, w\}$ of G', so a seperator $\{u, v, w\} = H_1 \cap H_2$ of G. Where $z_1 \in H_1$ and $z_2 \in H_2$ there are 3 internally disjoint z_1z_2-paths. Let

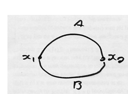


be the minors they yield. They are embeddable so can be embedded with z on the outside as shown, and so can be patched together. ◇

Now let xy be an edge of G and let $G' = G/xy$ where $v = v_{xy}$. Then G' has no K_5 or $K_{3,3}$ minor, so can be embedded. Further, by the claim, $G' - v = G - xy$ is 2-connected, so the face on which v lived was a cycle C.

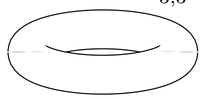


We expand v back to xy. Let X be teh neighbours of x on C and Y those of Y. If $|X| < 1$ or $|Y| < 1$ then the expansion is trivial, so assume that both contain at least 2 vertices. Also $|X \cap Y| \leq 2$ or we have a K_5 minor. If there exist x_1, y_1, x_2, y_2 in that order around the cycle, then there is a $K_{3,3}$ minor. So some x_1 and x_2 in X split C into paths A and B, such that $X \subset A$ and $Y \subset B$.



So there is an embedding.

□

Problem 7.8. Show that $K_{3,3}$ can be drawn without crossing edges on the surface of a torus: 

8 Colouring Planar Graphs

We saw that every graph G with $\chi(G) = k$ has a subgraph H with $\delta(H) = k-1$. So we get immediately that every planar graph is 6-colourable.

Problem 8.1. Show that any planar graph is 6-colourable.

In fact, more is true:

Theorem 8.1. *(The Four-Colour Theorem [Haken-Appel 1976]) Any planar graph is* 4*-colourable.*

The proof of the Four-Colour Theorem is very long and difficult. This is slightly easier, but still interesting:

Theorem 8.2 (Heawood 1890 (based on Kempe))**.** *Any planar graph is* 5*-colourable.*

Proof. (Sketch)

- Induction on $n = |V(G)|$, so we may assume that $n \geq 6$.
- By Euler's theorem there is a vertex v of degree at most 5, and by induction $G \setminus \{v\}$ has a 5-colouring, giving its neighbours $v_1, \ldots, v_5$, embedded cyclically, distinct colours.
- There is a $\{1, 3\}$-path between v_1 and v_3 or recolour á la Brooke's.
- With v this is a cycle separating v_2 and v_4.
- v_2 and v_4 cannot by in same component of $\{2, 4\}$-subgraph of G.

$\square$

Problem 8.2. This proof is based on an incorrect proof of the 4-colour Theorem by Kempe. Find the incorrect proof of the 4-colour Theorem on the internet.

Conjecture 8.3 (Hadwiger '43)**.** *Any k-chromatic graph has a K_k-minor.*

Hadwiger proved the case $k = 4$. The proof is difficult.

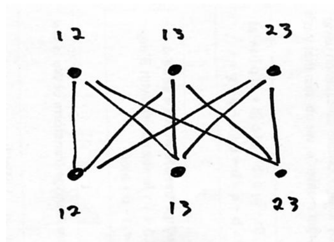
8.1 List Colourings

For a graph G a *set of lists* is a function $L : V(G) \to 2^{\mathbb{Z}}$. A function $\phi : V(G) \to \mathbb{Z}$ is a *list colouring* of (G, L) if it is a proper colouring of G and $\phi(v) \in L(v)$ for all $v \in V(G)$.

A graph G is k-list-colourable if there is a list colouring of (G, L) for any list L such that $|L(v)| = k$ for all v. The *list chromatic number* $\chi_L(G)$ of G is the minimum k for which G is k-list-colourable.

Clearly $\chi(G) \leq \chi_L(G)$ as one can take $L(v) = [k]$ for all v. And I know you think we have equality, because the lists are most restrictive when they are the same. But that isn't actually true.

Try to find a colouring of this set of lists for $K_{3,3}$



Problem 8.3.  Find a bipartite graph B with $\chi_L(B) = k$ for any $k > 2$.

Thomassen proved the following.

Theorem 8.4. *Every planar graph is* 5*-list colourable.*

Proof. We actually prove the following stronger statement. Let G be a plane graph in which every face, but possibly the outer face, is a triangle. Let $v_1 \sim v_2$ be an edge on the boundary B of the outer face. Then (G, L) has a list colouring for any set of lists L satisfying

- $|L(v)| \geq 5$ for $v \notin B$
- $|L(v)| \geq 3$ for $v \in B \setminus \{v_1, v_2\}$
- $|L(v_1)| = |L(v_2)| = 1$.

The proof is by induction on $n = |V(G)|$, and is clear if $n = 2$.

Case B isn't a cycle: Then there is a cut-vertex $u \in B$ separating G into G_1 and G_2. We may assume that v_1 and v_2 are in G_1. By induction G_1 has an L-colouring ϕ_1. On G_2, reduce the list $L(u)$ to $\phi_1(u)$ and the list of some neighbour in B to size 1. Then the list on G_2 meet the conditions above, so there is a colouring. This yields an L colouring of G.

Case B has a chord uw: Then $u \sim w$ is the intersection of a 2-separation $G_1 \cup G_2$ of G. Again we may assume that v_1 and v_2 are in G_1, and get that

G_1 has an L colouring ϕ_1 by induction. Reducing the lists of u and w in G_2 we still get a good set of lists, so can colour it.

Case B is an induced cycle: Let v_3 be the other neighbour of v_2 in B, and let $v_2, u_1, \ldots, u_k, v_4$ be a cyclic ordering of the neighbours of v_3. $L(v_3)$ contains two colours a and b not in $L(v_2)$. Removing v_3 from the graph and the colours a and b from the lists of $u_1, \ldots, u_k$ and v_4 (unless $v_4 = v_1$), we can list colour $G \setminus \{v_3\}$ by induction. Replacing v_3, possible v_4 has been coloured a or b, but one of them is available to colour v_3. □

8.2 Discharging

Discharging is a beautiful technique that has yielded many difficult results. We see how a simple example of it is used to give the following technical Lemma which is used in the in the proof of the 4-colour theorem.

Lemma 8.5. *Let G be a triangulation of the plane. Then one of the following is true*

i. G has a vertex of degree ≤ 4.

ii. G has two adjacent vertices of degree 5.

iii. G has a triangle with vertices of degrees 5 and 6.

Proof. Assuming i) is false we shot that ii) or iii) hold.

Start by assigning an initial charge $f_0(v) = 3(6 - \mathrm{d}(v))$ to every vertex v. By Euler's formula we have that

$$12 = 6n - 2m = \sum_V (6 - \mathrm{d}(v))$$

so $\sum_v f_0(v) = 36$ is the total initial charge.

Now, for every vertex v of degree 5, and every neighbour v' of v of degree at least 7, move a charge of 1 from v to v', and let $f_1 : V \to \mathbb{Z}$ be the final charge function. Clearly f_1 satisfies $\sum_V f_1(v) = 36$, so there is some vertex u with $f_1(u) > 0$. If u has degree 5, then since it started with charge 3, it has at most 2 neighbours of degree greater than 6. So either it has a neighbour of degree 5 or adjacent neighbours of degree 6.

Also u cannot have degree 6 as such vertices have intial and final charge 0.

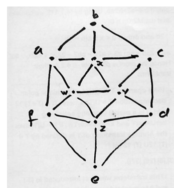
If u has degree 7, then it started with charge -3, so has at least 4 neighbours of degree 5, two of which must be adjacent.

If u has degree 8, then it stated with charge -6 so has at least 7 neighbours of degree 5, two of which must be adjacent.

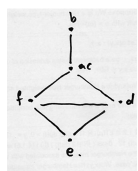
u cannot have degree 9 because then it had initial charge -9 and this could not be made positive with 9 neighbours. □

Using a similar Lemma, one can show that a planar graph must contain one of about 600 special subgraphs. One can than replace these with smaller subgraphs and use induction. One such examples is the following.

Problem 8.4. Let G be a planar graph containing the subgraph



show that it can be 4-coloured if the graph G' we get from it by replacing the above subgraph with the following one can be 4-coloured.



(We contract edges ax and xb, w, y and z, and add edge fd.)

9 Chordal Graphs and Lexicographic Breadth First Search

Colouring graphs– finding their chromatic number– is hard. But colouring trees is easy. Can we find other classes of graphs for which colouring is easy?

Note 9.1

lanar graphs are easy to 4-colour, and we can decide if a graph is planar, but Garey, Johnson and Stockmeyer showed in 1976 that it is hard to decide if a planar graph is 3-colourable.

Perfect graphs are a large class of graphs that are easy to colour. A graph G is perfect if for every induced subgraph $H \leq G$ we have that $\chi(H) = \omega(H)$. This doesn't apriori make colouring perfect graphs easy, because it is a hard problem to find the largest clique of a graph, but in 1988 Grötschel, Lovász, and Schrijver gave a polynomial time algorithm for finding the chromatic number of perfect graphs. This algorithm is a bit advanced though. Let's do something easier.

Recall that $\omega(H)$ is the clique number– order of the maximum clique.

Problem 9.1. Give an example of a graph that is not perfect (easy). Can you find a non-perfect 4-chromatic graph? (Harder. If you can't think of one, look at the Myceilski construction in Section 10.)

We have a good algorithm for colouring graphs– the greedy algorithm gives the chromatic number *if we start with the right ordering*. So maybe there are graphs for which we can easily find this ordering.

9.1 Interval graphs

Note 9.2

ecall:
The *eccentricity* of r in connected G is the maximum distance of an vertex from r in G. The i^{th} *distance neighbourhood* $N^i(r) = N^i_G(r)$ of a vertex r in a graph G is the set of vertices that are distance exactly i from r in G. So $N^0(r) = r$ and $N^1(r) = N(r)$, and the sets $N^i(r)$ for $i = 0, 1, \ldots, d$ where d is the eccentricity of r in G, is a partition of $V(G)$.

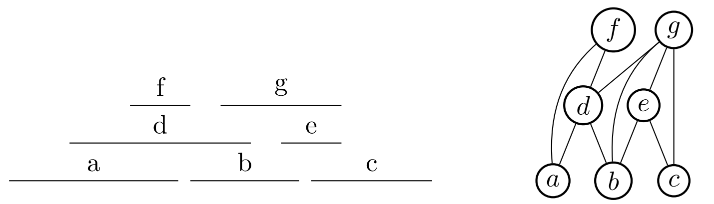
'Intersection graph' is a catch-all term for a graph whose vertices are mathematical (often geometric or set) objects such that two are adjacent if the intersect. Depending on the construction these graph can be quite natural – occuring in various applications, and have a lot nice properties.

A graph G is an *interval graph* if one can represent it as the intersection graph of intervals on the real line. That is if there is a set

$$\mathcal{I} = \{I_v \mid v \in V(G)\}$$

of closed intervals such that $u \sim v$ if and only if $|I_u \cap I_v| = \emptyset$.

Example 9.1. A graph and its interval representation. We draw the intervals not on the real line, but above them, so that we can see them all.



Problem 9.2. Show that (for finite graphs) we can replace the 'closed' in the definition of interval graph with 'open' or by nothing (no restriction on the intervals. Show that we may assume no two intervals share an endpoint.

There are two common restrictions on the intervals allow for an interval graph. An interval graph is a

- *unit* interval graph if all intervals can be taken to have unit length.
- *proper* interval graph if no interval properly contains another interval

Problem 9.3. Find an interval graph that is not a unit interval graph. Show that a graph is a proper interval graph if and only if it is a unit interval graph.

A *k-hole* in a graph G is an induced copy of the cycle C_k in G. A*k-antihole* in G is an induced copy of C_k in $\overline{G}$.

Problem 9.4. Show that any graph with a 5-hole also contains a 5-antihole.

The following gives a very useful propery of interval graphs, we will come back to it.

Problem 9.5. Show that an interval graph has no k-holes for $k \geq 4$. Show that interval graphs may have 4-antiholes, but have no k-antihole for $k \geq 5$. (Hint: The proof of the last statement for $k = 5$ is basically done above, and for $k \geq 6$ can be done at once in just a line or two.)

Having the interval representation of an interval graph makes several difficult problems easy.

Problem 9.6. Show that a set of vertices in an interval graph induces a clique if and only if the associated set of intervals has non-empty intersection. Explain how you would therefore find the maximum clique of an interval graph, if you had its interval representation.

Problem 9.7. Order the vertices of an interval graph by the ordering of the left endpoints of their intervals, (we showed above these can be assumed to be distinct). Show that a greedy colouring uses $\omega(G)$ colours. Conclude that $\chi(G) = \omega(G)$ for interval graphs.

A graph class is *hereditary* if G being in the class and H being an **induced** subgraph of G implies that H is also in the class.

Problem 9.8. Show that the class of interval graphs is hereditary, conclude, using Problem 9.7, that interval graphs are perfect.

Okay. So having this interval represenation of a graph is nice. How do we find it. One can do so in linear time, but we look at chordal graphs first.

9.2 Chordal Graphs

Interval graphs have this nice property that they have no *holes*. This property arises everywhere. A graph G is *chordal* if it has no holes. So interval graphs are chordal.

Chordal graphs can also be viewed as intersection graphs.

Problem 9.9. Let T be a tree and $\mathcal{T}$ be a collection of subtrees of T. Show that the intersection graph of $\mathcal{T}$ (each subtree in $\mathcal{T}$ is a vertex and two are adjacent if they share a vertex) is chordal.

In fact one can show that a graph is chordal if and only if it has a representation, as in Problem 9.9 as the intersection graph of a family of subtrees of a tree. This is a little hard to show without some work; but clearly for interval graphs, the tree T in the subtree representation can be taken to be a path. With this in mind:

Problem 9.10. Find a chordal graph that is not an interval graph.

There are all sorts of different characterisation of chordal graphs. We look at one more that will help us decide if a graph is chordal.

An vertex v of a graph G is *simplicial* if its neighbours induce a clique. An ordering $v_1, \ldots, v_n$ of the vertices of a graph G is a *simplicial ordering* or a *perfect elimination ordering* if for each i the vertex v_i is simplicial in the subgraph G_i of G induced by the vertex set $\{v_1, \ldots, v_i\}$.

Problem 9.11. Show that if v is simplicial in G and $G' = G \setminus \{v\}$ is chordal, then G is chordal. Conclude that any graph with a simplicial ordering is chordal.

Problem 9.12. Show that interval graphs have simplicial orderings.

The following should not be so much harder than the similar results for interval graphs.

Problem 9.13. Given a simplicial ordering of a graph G, explain how you would find the largest clique in G. Show that chordal graphs are perfect.

Dirac, in 1962, showed the following.

Theorem 9.2. *A graph is chordal if and only if it has a simplicial ordering.*

The difficult part of this is showing that any chordal graph has a simplicial vertex. We do this with the following stronger statement. (This proof is from [4] and is attributed to Voloshin and Farber-Jamison.)

Lemma 9.3. *Let x be a vertex in a chordal graph G. Then among the vertices of G that are furthest from x, there is a simplicial vertex of G.*

Proof. The proof is by induction on n and is trivial if $n = 1$, so assume that $n \geq 2$. If x is adjacent to every other vertex, then by induction there is a simplicial vertex in $G \setminus \{x\}$, and this vertex is simplicial in G. So we may assume that the eccentricity d of x is at least 2.

Let $T = N^d_G(x)$ and let H be a component of the subgraph $G[T]$ of G induced by T. Let S be the set of vertices in N^{d-1}_G with neighbours in H.

We claim that S is a clique. Indeed, if there were non-adjacent vertices s and s' in S then there is an induced path P between them in H, and an induced path P' between them in $G(V) \setminus (N^d(x) \cup N^{d-1}(x))$. There can be no edges between P and P' so with s and s' they make an induced cycle of girth at least 4. But this is impossible.

By induction, taking $x = s$ the graph $G' = G[S \cup V(H)]$ contains a simplicial vertex z that is not adjacent to x, so is in H, unless G' is a clique, in which case we can take z to be in H. As the neighbours of z are the same in G and G', it is simplicial in G. □

Problem 9.14. Use Lemma 9.3 to prove Theorem 9.2.

Problem 9.15. Use the fact that every chordal graph G has a simplicial vertex to show that every chordal graph has a subtree representation in which the vertices of the tree T are the maximal cliques of G, and the subtree T_v for a vertex v of G consist of all cliques that v is in.

9.3 Breadth First Search

Many algorithms on graphs start with an ordering $v_1, \dots v_n$ of the vertices. There a many ways to order the vertices of a graph, and one of the most common is the BFS ordering. The 'BFS' comes from a 'breadth first search', and refers to a spanning tree T of the graph that we may construct in constructing the ordering.

To defined the construction of the BFS ordering we us the algorithmic notion of a queue (or list). This is simply an ordered set to which we can add or remove elements. When we remove elements we removed minimum elements (we remove the from the start of the list) and when we add them we add maximum elements (we add them to the end of the list).

The ordering of $V(G)$ produced by the following algorithm is a BFS-*ordering from* v_1

Algorithm 9.4 (BFS(G, v_1))**.**

Input: *A connected graph* G *and a vertex* $v_1 \in V(G)$.
Ouput: *An ordering* $v_1, \dots, v_n$ *of* $V(G)$.
Initialisation: *Let* $i = 1$, Q *be a queue containing* $N(v_1)$ *and* R *be the set* $V(G) \setminus (\{v_1\} \cup N(v_1))$
Algorithm: *While* Q *is not empty, increase* i *by* 1 *and:*

i. Remove a vertex from Q *and call it* v_i.

ii. Add the neighbours of v_i *in* R *to* Q*, and remove them from* R.

Problem 9.16. Show that a BFS-ordering from r satisfies the following:

$$i \leq j \Rightarrow d(r, v_i) \leq d(r, v_j).$$

(The closer a vertex is to r the earlier we visit it.)

Problem 9.17. Draw a reasonably random looking graph on 10 vertices with around 25 edges, label its vertices, and find a BFS ordering of the vertices.

Problem 9.18. Write a program in Python or Sage to find a BFS ordering of an inputted graph and starting vertex. Prove that its running time is $O(|n| + |m|)$.

Problem 9.19. Show by example that BFS(G, v) can produce different orderings of $V(G)$.

Problem 9.20. Show that finding the distance between two vertices u and v of G can be done with one appliction of BFS. Show that the diameter of a graph can be determined in polynomial time.

Problem 9.21. Explain how to use BFS on a graph G to decide if it is bipartite.

BFS orderings of a graph can be quite useful, but if we are more discerning about how we order that vertices are added to Q, they can be even more useful.

9.4 Lexicographic Breadth First Search

We will now give a more deterministic algorithm for finding a BFS-ordering of $V(G)$. Before we give our algorithm, we need to define some notation. The queue Q is an ordering. This induces an ordering $\leq_Q$ on the subsets of Q by setting $A \leq_Q B$ if $A \setminus B$ contains a smaller (in Q) element than $B \setminus A$ does. (This is called a lexicographic ordering of subsets of Q.)

The following is the *lexicographic breadth first search*.

Algorithm 9.5 (LBFS(G, v_1))**.**

Input: *A connected graph G and a vertex $v_1 \in V(G)$.*
Ouput: *An ordering $v_1, \ldots, v_n$ of $V(G)$.*
Initialisation: *Let $i = 1$, Q be a queue containing $N(v_1)$ and R be the set $V(G) \setminus (\{v_1\} \cup N(v_1))$*
Algorithm: *While Q is not empty, increase i by 1 and:*

i. Remove a vertex from Q and call it v_i.

ii. Add the neighbours of v_i in R to Q, in the following order: add a before b if

$$N(a) \cap Q \leq_Q N(b) \cap Q.$$

iii. Remove neighbours of v_i from R.

Problem 9.22. Show that an ordering produced by LBFS(G, v_1) is a BFS-ordering.

Problem 9.23. Show (again) that LBFS(G, v) can produce different orderings of $V(G)$.

Theorem 9.6. *A graph G is chordal if and only if* BFS(G, v_1) *produces a simplicial ordering for some choice of v_1, if and only if it produces a simplicial ordering for every choice of v_1.*

Proof. We have seen that if G has a simplicial ordering then it is chordal, so it is enough to show that if G is chordal, and $\sigma = (v_1, \ldots, v_n)$ is output by LBFS(G, v_1), then σ is simplicial. By induction, it is enough to show that if G is chordal then v_n is a simplicial vertex of G. As LBFS(G, v_1), produces a BFS-ordering, we have that $d = d(v_1, v_n)$ is the eccentricity of v_1. It has neighbours in $N^{d-1}(v_1)$ and possibly also in $N^d(v_1)$. To see that v_n is simplicial we have to show that any two neighbours v_i and v_j of v_n are adjacent. There are three cases. (We write N^i for $N^i(v_1)$.)

i. $v_i, v_j \in N^{d-1}$.

ii. $v_i \in N^{d-1}$ and $v_j \in N^d$.

iii. $v_i, v_j \in N^d$ wlog $i < j$.

Case (i) is by (essentially) the proof of Lemma 9.3: using a minimal path P between v_i and v_j such that all the vertices but v_i and v_j are distance at most $d-2$ from v_1, we get with $v_i \sim v_n \sim v_j$ an induced cycle of length at least 4, unless $v_i \sim v_j$. Thus we conclude that $v_i \sim v_j$.

For case (ii) we are done if $N(v_j) \cap N^{d-1} = N(v_n) \cap N^{d-1}$, so we may assume they are not. In this case, as v_n was the last vertex v_j has a neighbour v_i' in N^{d-1} that is not adjacent to v_n. Now the same proof works.

Case (iii) is harder. To make things clearer we write $x \leq y$ to mean x comes before y in the ordering σ. Before we get started, observe that if $u \leq v \leq w$ and $w \sim u \not\sim v$ then there must be some $u' < u$ such that $w \not\sim u' \sim v$– if not, then w would have be chosen before v by the LBFS. Let $F(v, w)$ be σ-minimum vertex u' that satisfies this. By minimality we now have for all $x < F(v, w)$ that $(x \sim v) \iff (x \sim w)$.

Now lets get started with case (iii). We relabel vertices to make notation easier: assume that a_0 is the σ-maximum element and that it has neighbours a_1 and b_1 in N^d with $a_1 \leq_\sigma b_1$. We want to show that $a_1 \sim b_1$ so towards contradiction assume $a_1 \not\sim b_1$. Let $b_2 = F(b_1, a_0)$. Thus $b_2 \leq a_1$, and $b_1 \sim b_2 \not\sim a_0$. If $a_1 \sim b_1$ then we have an induced C_4, so we may assume that the path a_1, a_0, b_1, b_2 is induced. Let $a_2 = F(a_1, b_1)$. Thus $a_2 \leq b_2$, and $a_2 \sim a_1 \not\sim b_1$. As $a_2 < F(b_1, a_0)$ we also have $a_2 \not\sim a_0$. If $a_2 \sim b_2$ then we have an induced 5-cycles, so we may assume that $a_2 \not\sim b_2$, and so the path a_2, a_1, a_0, b_1, b_2 is induced. Continuing in this way we eventually get an induced path $a_i, \ldots, a_2, a_1, a_0, b_1, b_2, \ldots, b_i$ such that neither a_i or b_i are in N^d. From here, we finish as in the previous cases.

□

With this we can test if G is chordal by picking a vertex v_1, running BFS(G, v_1) and then checking if the resulting ordering is simplicial.

Problem 9.24. Use this test to show that the Petersen graph is not chordal. (Or pick another reasonably complicated graph on 10 vertices and test its chordality by this test.)

Problem 9.25. Program a chordality test for a given graph G.

Problem 9.26. How long does the test for chordality take?

Notice that the ordering produced by LBFS(G, v_1) is still not unique– two vertices in U might have the same minimum label set. To make it unique, we have to decide how ties are broken. We can do this with the following variation that uses an initial ordering $\sigma = (u_1, \ldots, u_n)$ of $V(G)$.

Algorithm 9.7 (LBFS(v_1, σ))**.**

Input: *A connected graph G and a vertex $v_1 \in V(G)$.*
Ouput: *An ordering $v_1, \ldots, v_n$ of $V(G)$.*
Initialisation: *Let $i = 1$, Q be a queue containing $N(v_1)$ and R be the set $V(G) \setminus (\{v_1\} \cup N(v_1))$*
Algorithm: *While Q is not empty, increase i by 1 and:*

i. *Remove a vertex from Q and call it v_i.*

ii. *Add the neighbours of v_i in R to Q, in the following order: add a before b if*
$$N(a) \cap Q \leq_Q N(b) \cap Q,$$
or if $N(a) \cap Q = N(b) \cap Q$ and $a \leq_\sigma b$.

iii. *Remove neighbours of v_i from R.*

We will not go into the details, but to check if a graph is an interval graph one can run LBFS(v_1, σ) six times, but after the first time, you must use the results of earlier runs, some times with the ordering reversed. It seems like magic.

10 Graph Homomorphsims

See the book **Graphs and Homomorphisms** by Hell and Nešetřil [3] for this material.

Given two graphs G and H, a homomorphsim $\phi : G \to H$ is a vertex mapping $\phi : V(G) \to V(H)$ that preserves edges:

$$u \sim_G v \Rightarrow \phi(u) \sim_H \phi(v).$$

The language of homomorphsims is useful for much graph terminology:

- A walk of length n in a graph H is a homomorphism of a P_n to H.
- It is a trail if the homomorphism is edge-injective and a path if the homomorphism is vertex-injective.
- A k-colouring of a graph G is a homorphism of G to K_k.
- A graph is bipartite if and only if it admits a homomorphism to K_2.
- A graph is bipartite if and only if it admits no homomorphsim from an odd cycle.

Problem 10.1. Show that the relation $G \to H$ is a transitive reflexive relation on the class of all graphs, but not asymmetric.

Two graphs G and H are *homomorphically equivalent* if $G \to H$ and $H \to G$.

Problem 10.2. Let $[H]$ be the set of graphs that are homomorphically equivalent to a graph H. Show that, upto isomorphism, there is a unique smallest (wrt number of vertices) graph C in $[H]$. This graph C is the *core* of H.

Homomorphism defines a partial order on the family of cores: $G \leq H$ if $G \to H$.

Problem 10.3. Does the graph homomorphism order have a maximum or minimum element. (We consider only finite graphs.)

Clearly

$$K_1 \to K_2 \to K_3 \to K_4 \to \cdots$$

is an infinite ascending chain in this order and

$$\cdots \to C_{11} \to C_9 \to C_7 \to C_5 \to C_3$$

is an infinite descending chain.

An *antichian* in the homomorphism order is a set of graphs amongst which there are no homomorphisms.

Theorem 10.1 (Mycielski 1955)**.** *For $k \geq 3$ there is a k-chromatic graph containing no K_3.*

Proof. For a graph G let the Mycielski graph $M = M(G)$ be the graph defined as follows.

$$V(M) = \{w_0\} \cup \bigcup_{v_i \in VG} \{v_i, u_i\}$$

$$E(M) = \bigcup \{u_i v_j, v_i v_j \mid v_i v_j \in E(G)\} \cup \{w_0 u_i \mid v_i \in V(G)\}.$$

Problem 10.4. Show that if G is K_3-free then $M(G)$ is and that $\chi(M) = \chi(G) + 1$

The proof follows by induction, starting with $G = C_5$.

□

Problem 10.5. Alter Mycielski's construction to show that for every odd $g \geq 3$ there is a g-chromatic graph with odd girth $2g + 1$.

Problem 10.6. Show that there are infinite antichains in the homomorphism order.

A pair (A, B) is a *gap* in the homomorphism order if $A < B$ and there is no graph C such that $A < C < B$.

Problem 10.7. Show that (K_1, K_2) is a gap.

We will show that (K_1, K_2) is the only gap in the homomorphism order. This also follows from Problem 10.5 but uses a couple more nice little ideas.

Mycielski's result can be strengthened as follows. The proof uses the probabilistic method; we will see it later.

Theorem 10.2 (Erdős 1959)**.** *For all $g, k \geq 3$ there exist graphs of girth g and chromatic number k.*

10.1 Products and the Exponential Graph

The *(categorical) product* $G \times H$ of two graphs G and H is defined by $V(G \times H) = V(G) \times V(H)$ and

$$(g, h) \sim (g', h') \iff g \sim g' \text{ and } h \sim h'.$$

Problem 10.8. Show that $G \times H$ has the following property, (which means that it is the *product* in the category of graphs).

For any graph R with homomorphisms $g : R \to G$ and $h : R \to H$, there is a unique homomorphism $f : R \to G \times H$ such that $g = \pi_1 \circ f$ and $h = \pi_2 \circ f$.

A graph K is *multiplicative* if $G \times H \to K$ implies that $G \to K$ or $H \to K$.

Hedetniemi conjectured that all complete graphs are multiplicative:

Conjecture 10.3 (Hedetniemi)**.** *If the product $G \times H$ is k-colourable, then one of its factors G or H is k-colourable.*

This was proved for $k = 3$ by El-zahar and Sauer in 1985. The proof is quite difficult. The conjecture is still open for $k \geq 4$.

Problem 10.9. Rödl and Pudlack showed that the Hedetniemi conjecture is not true of we allow G and H to be digraphs. Let T be the transitive tournament on 4 vertices; that is $V(T) = \{1, \ldots, 4\}$ and $i \to j \iff i < j$; and let S be T with the arc $(1, 4)$ reversed. Viewing K_3 as the digraph on 3 vertices with 6 arcs, show that $T \not\to K_3$ and $S \not\to K_3$ but $T \times S \to K_3$.

For graphs G and H, let the *exponential graph* H^G has as its vertices the set of mappings (not homomorphsims) of $V(G)$ to $V(H)$. Two maps ϕ and ϕ' are adjacent if $\phi(g) \sim \phi'(g')$ for all edges $g \sim g'$ of G.

Proposition 10.4. *For graphs F, G and H we have*

- $H^{G+F} \cong H^G \times H^F$
- $H^{G \times F} \cong (H^G)^F$
- $F \times G \to H$ *if and only if* $F \to H^G$.

Proof. i) We claim that $\Phi : H^{G+F} \to H^G \times H^F$ defined for a map $\phi \in H^{G+F}$ by $\Phi(\phi) = (\phi|_G, \phi|_F)$ is an isomorphism. It is clearly a bijection as $\phi \neq \phi'$ if and only if ϕ and ϕ' differ on some vertex g of G or f of F, if and only if $\phi|_G$ differs from $\phi'|_G$ or $\phi|_F$ differs from $\phi'|_F$.

To see that it is an isomorphism show that if $\phi \sim \psi$ in H^{G+F} if and only if then $\phi_G \sim \psi_G$ in H^G and $\phi_F \sim \psi_F$ in H^F. Both are true if and only if $\phi_G(g) = \phi(g) \sim \psi(g') = \psi_G(g')$ for every $g \sim g'$ in G and blah blah for F.

ii) Exercise (Or see Hell Nešetřil.)

iii) This is immediate from ii) as $H^{G \times F}$ has a loop if and only if $(H^G)^F$ does. □

Theorem 10.5. *The only gap in the homomorphism order is* (K_1, K_2).

Proof. We must show that if $(G, H) \neq (K_1, K_2)$ and $G \to H$ (but $G \not\leftarrow H$), then there is a graph M such that $G \to M \to H$ (but $G \not\leftarrow M \not\leftarrow H$).

The graph $M = G + (Z \times H)$ will do, where Z is a graph, from Problem 10.5, of odd girth greater than the odd girth of G having chromatic number greater than the chromatic number of G^H.

Indeed, G maps to the copy of G in M, and as $G \to H$ and $Z \times H \xrightarrow{[} \pi_2]H$, we have that $M \to H$. On the other hand as Z has large odd girth, so does $Z \times H$, and so $H \not\to Z \times H$, we also have $H \not\to G$, so $H \not\to M$. Finally, if $M \to G$, then in particular $Z \times H \to G$ and so $Z \to G^H$; but Z has larger chromatic number than G^H, so this is impossible.

□

10.2 H-Colouring Dichotomy

For a graph H the problem of *H-colouring* denoted $\mathrm{Hom}(H)$ is to decide for a given instance G whether or not there is a homomorphism $G \to H$.

As 3-colouring is difficult, or NP-complete, we have that $\mathrm{Hom}(K_3)$ is NP-complete.

Problem 10.10. Show that $\mathrm{Hom}(H)$ can be decided in polynomial time.

What about for other graphs? Hell and Nesětřil showed the following in 1990.

Theorem 10.6. *For any H containing an odd cycle,* $\text{Hom}(H)$ *is NP-complete.*

The proof of this uses a series of polynomial reductions.

Example 10.7 (Proof that C_5 is in NPC)**.** Let C_5 have vertices $1, \ldots, 5$ with vertices being adjacent if they are consecutive mod 5. Let P be a path on the four vertices $v_1, \ldots, v_4$ with vertices being adjacent if their indices are consecutive.

Given an instance G of $\text{Hom}(K_5)$, let the graph G^* be constructed from G by the following construction. For every original edge uv of G, remove uv and add a new copy of P by identifying the vertices v_1 and v_4 in the copy of P with the vertices u and v of G respectively.

Notice that there for $i \neq j \in \{1, \ldots, 5\}$ there are C_5-colourings ϕ_{ij} of P such that $\phi_{ij}(v_1) = i$, and $\phi_{ij}(v_4) = j$; however, there is no C_5-colouring of P that takes both v_1 and v_4 to the same image. One can thus verify that

$$G^* \to C_5 \iff G \to K_5.$$

Thus $K_5 <_{NP} C_5$, and so because K_5 is in NPC, so is C_5.

Problem 10.11. Using a similar indicator construction as in the example above, prove that if $\text{Hom}(H)$ is NP-complete for all graphs containing a triangle, then it is true for all graphs containing an odd cycle.

The rest of the proof of Theorem 10.6 takes some work. But it follows easily from a (much stronger) result of Barto and Kozik which holds for all relational structures H.

A homomorphism $\phi : H^d \to H$ is a *cycle term* if

$$\phi(v_1, \ldots, v_n) = \phi(v_2, \ldots, v_n, v_1) = \cdots = \phi(v_n, v_1, \ldots, v_{n-1}).$$

Theorem 10.8. *If H has no cycle term of arity p, for any given prime $p > |V(H)|$, then* $\text{Hom}(H)$ *is NP-complete.*

Now let H be a simple graph containing an odd cycle. Let p be an odd prime greater than n. By the theorem, if we can show that H has no cycle term or arity p, then $\text{Hom}(H)$ is NP-complete. Assume it has a p-cycle term.

Problem 10.12. Show that if H has an odd cycle of length $g \leq p$ for p odd, then it has a closed circuit $v_1 \sim v_2 \sim \ldots sim v_p \sim v_1$ of length p.

But then

$$\phi(v_1, \ldots v_p) = \phi(v_2, \ldots, v_p, v_1)$$

and

$$\phi(v_1, \ldots v_p) \sim \phi(v_2, \ldots, v_p, v_1)$$

which means that H contains a loop. But this is a contradiction.

The converse of the Theorem 10.8 is an open conjecture.

Conjecture 10.9 (CSP-dichotomy conjecture)**.** *If a digraph (or any relational structure) H has a cycle term, then* $\mathrm{Hom}(H)$ *is polynomial time solvable.*

We know this is true if H has a totally symmetric idempotent ϕ:

$$\phi(v_1, \ldots, v_n) = \phi(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$$

for all permutations σ of $[n]$.

11 Extremal Graph Theory

The basic idea of Extremal Graph Theory is to find the largest (smallest) graph with (without) a given property. Largest (smallest) can refer to the number n of vertices or m of edges, but often n is fixed, and largest refers to m.

11.1 Turan's Theorem

For a graph H, the *extremal number* $\mathrm{ex}(n, H)$ for H is the maximum number of edges a graph on n vertices can have without containing a copy of H as a subgraph.

The Turan graph $T_{k-1}(n)$ is the complete $(k-1)$-partite graph with vertex partition

$$V = V_1 \cup V_2 \cup \ldots V_{k-1}$$

where

$$|V_1| \leq |V_2 \leq \cdots \leq |V_{k-1}| \leq |V_1| + 1.$$

When $n = q(k-1)$ for integer q, this has $\binom{k-1}{2}q^2 = \frac{k-2}{2(k-1)}n^2$ edges.

Problem 11.1. Find a general formula for the size $t_{k-1}(n)$ of $T_{k-1}(n)$.

Theorem 11.1. *[Turan 1941] For all $n \geq k > 1$, $\mathrm{ex}(n, K_k) = t_{k-1}(n)$.*

Proof. It is enough to show that the maximum number of edges in a graph omitting K_k is a complete multipartite– by convexity the turan graph has the most edges for a fixed number of parts, and the edges count increases in the number of parts, which is bounded by $k-1$, so this will do.

Let G be an extremal K_k-free graph on n vertices. If it is not complete multipartite, then there are adjacent vertices y_1 and y_2 with common non-neighbour x. If $d(y_1) > d(x)$, then deleting x and replacing it with a clone of y_1 gives a K_k-free graph with more edges; a contradiction. On the other hand, if $d(y_1), d(y_2) \leq d(x)$, then removeing y_1 and y_2 and cloning x twice gives a K_k-free graph with more edges. □

Problem 11.2. Show that when we add an edge to $T_{k-1}(n)$ we get no only one copy of K_k, but at least $\lfloor \frac{n}{k-1} \rfloor^{k-2}$ of them.

This is not only true of the Turan graphs.

Theorem 11.2. *[Erdős-Stone '46] For all integers $k \geq 2$ and $s \geq 1$, and every $\varepsilon > 0$ there exists an integer n_0 such that every graph with $n \geq n_0$ vertices and at least $t_{k-1}(n) + \varepsilon n^2$ edges, contains a $K_{s,s,\ldots,s}$ (with k 's's).*

We will prove this later with the regularity lemma.

11.2 Ramsey Theory

Ramsey showed the following.

Theorem 11.3. *For all $s, t \geq 3$ there exists a number n such that for any red-blue colouring of the edges of K_n there is either a red K_s or a blue K_t.*

The *ramsey number* $r(s,t)$ is the minimum such n. The ramsey numbers that we know are:

s,t	3	4	5
3	6		
4	9	18	
5	14	25	$43-49$
6	18	$35-41$	$58-87$
7	23	$49-61$	$80-143$

Problem 11.3. Show that for all $t \geq 3$, $r(t,t) < 2^{2t}$.

The proof of the following theorem is one of the early uses of the probabilistic method, which we will see more of, in the next section.

Theorem 11.4 (Erdős)**.** *For all $t \geq 3$ we have $r(t,t) > 2^{t/2}$*

Proof. Red-blue colour the edges of K_n as by letting each edge (independently) be blue with probability 1/2. For a random set S of vertices, the probability that S induces a mono-chromatic K_t is

$$2 \cdot \left(\frac{1}{2}\right)^{\binom{t}{2}} = 2^{1-\binom{t}{2}}.$$

There are $\binom{n}{b}$ such sets, so the probability that at least one is monochromatic is less than

$$\binom{n}{t}\frac{2}{2^{\binom{t}{2}}} < \frac{n^t}{t!}\frac{2}{2^{\binom{t}{2}}} = \frac{n^t}{2^{t^2/2}}\frac{2^{1+t/2}}{t!} < \frac{n^t}{2^{t^2/2}}.$$

Setting $n = 2^{t/2}$, this is less than 1, so some random colouring of K_n yields no mono-chromatic K_t. □

More generally we can define $R(H_r, H_b)$ for two graphs H_r and H_b as the minimum n such that for any red-blue colouring of the edges of K_n there is a red copy of H_r or a blue copy of H_b.

This is clearly equivalent to finding the minimum n such that for any graph G on n vertices, either G contains a copy of H_r or $\overline{G}$ contains a copy of H_b.

Problem 11.4. For a tree T on t vertices, find $R(T, K_s)$.

We will show the following, once we have seen the regularity lemma.

Theorem 11.5 (Chvatal, Rödl, Szemeredi, Trotter '83)**.** *For all Δ there exists $c_\Delta > 0$ such that for all H with $\Delta(H) \leq \Delta$ we have $R(H, H) < c_\Delta |V(H)|$.*

11.3 VanDerWaerden

The *Van Der Waerden number* $W(2, \ell)$ is the smallest n such that an 2-colouring of $[n]$ yields a monochromatic arithemetic progression of length ℓ. Van Der Waerden proved such numbers exist, in 1927. (They still exist.)

Problem 11.5. Show that $W(2, 5) < 325$.

The best known bound for $W(k, \ell)$ is

$$W(k, \ell) < 2^{2^{k^{2^{\ell+9}}}}$$

by Gowers. Graham offers \$1000 for showing

$$W(2, \ell) < 2^{k^2}.$$

12 Probabilistic Method

We saw one proof using the probabilistic Method in Section 11.2. Proofs like this are simply counting proofs.

Consider the following two proofs of a simple result. They are essentially the same proof.

Proposition 12.1. *Every graph G has a bipartite subgraph H with $m_H \geq m_G/2$.*

Counting proof. Count the number P of pairs (S, e) where S is a subset of $V(G)$ and $e = \{u, v\}$ is an edge of G with one endpoint in S. We choose the edge, then endpoint, and then the other elements of S; so

$$P = m_G \cdot 2 \cdot 2^{n-2} = m_G 2^{n-1}.$$

There are 2^n different subsets S, so one is counted in at least $m_G/2$ pairs. H is the bipartite subgraph of G induced by the partite sets S and $V(G) \setminus S$. □

Probabilistic Version. Randomly choose a subset S ov $V(G)$ by putting each vertex v of G in S with probability $1/2$. Let H be the bipartite subgraph of G induced aby partite sets S and $V(G) \setminus S$. We show that the expected number of edges in H is at least $m_G/2$. Indeed, each edge of G is in H with probability $1/2$. So the expected number of edges in H is $\sum_{e \in G} 1/2 = m_G/2$.

It follows that there is some choice of S so that the induced subgraph H has at least $m_G/2$ edges. □

12.1 Random Graphs

The *Erdős Rényi Random Graph* $G_{n,p}$ on n vertices $[n]$ is constructed as follows. For each pair of vertices u, v the edge (u, v) is in $G_{n,p}$ with probability p, independent of the existence of other edges.

Formally, the random graph $G_{n,p}$ as a probability space, containing the $2^{\binom{n}{2}}$ (labelled) graphs on n vertices. The probability that $G_{n,p}$ is a given graph H on the vertices $[n]$ depends on the number of edges of H. If H has $|E(H)| = m$ edges, then the probability that $G_{n,p} = H$ is $p^m(1-p)^{\binom{n}{2}-m}$.

Computing the probability that $G_{n,p}$ contains an isomorphic copy of a certain subgraph is a little harder. But a starting point is to look at the expected number of copies of a subgraph.

Technically, we define a random variable $X_H = X_H(G_{n,p})$ that counts the copies of a subgraph H, and use the linearity of expectation to find its expected value.

The expected number of edges, or copies of K_2 in $G_{n,p}$ is $\sum_{u \neq v \in [n]} p = p \cdot \binom{n}{2}$. To find the expected number of k-cycles we observe that there are $n!/2k(n-k)!$ possible k-cycles on $[n]$ and that each occurs in $G_{n,p}$ with probability p^k. So the expected number of k-cycles is

$$n!/2k(n-k) \cdot p^k < (np)^k/k2$$

More generally, the expected number of cycles of length less than or equal to k is

$$\sum_{i=3}^{k} \frac{n^i p^i}{2i} < (k-2)(np)^k.$$

Note

f(x) = o(g(x)):

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$

Now think of p as a function of n. If $p < n$, that is, $\lim_{n \to infty} p/n < 1$ then the expected number C of cycles of length k or less is $o(1)$.

Markov's inequality allows use to use expected value to bound the probability of an event:

For any positive random variable X

$$Prob(X \geq a) \leq E(X)/a,$$

where $E(X)$ is the expected value of X.

Apply this to the RV X that counts the cycles in $G_{n,p}$. If $p < n$ then the expected number of cycles $E(X)$ is less than ε for large enough n, so Markov's inequality gives us that the probability that $G_{n,p}$ has a cycle, it that $X \geq 1$ is less than ε.

That is, when $p < n$, $G_{n,p}$ is *asymptotically almost surely (aas)* acyclic: the probability that $G_{n,p}$ is acyclic goes to 1 as n goes to ∞.

Problem 12.1. Show that when $p = 1/n$ the expected number of cycles of any length is is less than $\ln(n/2)$.

Problem 12.2. Show that if $1/n = o(p)$ then $G_{n,p}$ has a cycle asymptotically almost surely.

Note 12.1

n fact, letting p increase from 0 to 1 we can look at $G_{n,p}$ as evolving from the empty graph on n vertices to the complete graph. Some epochs of the evolution are as follows.

- If $p = o(1/n)$ then $G_{n,p}$ is asymptotically almost surely a forest.
- If $p = c/n$ for $c < 1$ then $G_{n,p}$ is aas a union of single cycle components.
- If $p = c/n$ for $c > 1$ then $G_{n,p}$ ass has a giant component of $\Theta(n^{2/3})$ vertices.
- If $p = c \log n/n$ then $G_{n,p}$ is aas connected.

Now, we prove Theorem 10.2, which says that there is a graph of girth at least g and chromatic number at least k.

Proof of Theorem 10.2. The main idea is to choosing p so that $G_{n,p}$

- has as most $n/2$ cycles of length at most k, with probability less than $1/2$
- has an independent set of size at least $n/2k$, with probability less than $1/2$.

It follows that there exists a graph on n vertices with both of these properties. Then we can kill all cycles of length k or less by removing at most $n/2$ vertices. The remaining graph has at least $n/2$ vertices and independent set of size at most $n/2k$, so has chromatic number at least k. This is enough.

Let $k \geq 3$ be given. To choose p, fix some ε with $0 < \varepsilon < 1/k$ such that

$$\frac{6k \ln n}{n} < \frac{n^{\varepsilon}}{n} < \frac{n^{1/k}}{n},$$

and let $p = n^{\varepsilon}/n$.

We first show that the probability that the independence number α of $G = G_{n,p}$ is greater than $n/2k$ goes to 0 as n goes to ∞. Indeed, the probability thah $G_{n,p}$ has an independent set of size t (or greater) is less than

$$\binom{n}{t}(1-p)^{\binom{t}{2}} < \left(n(1-p)^{(t-1)/2}\right)^t \leq (ne^{-p(t-1)/2})^t$$

where the last inequality uses that $1 - x < e^{-x}$ for $x > 0$.

Using, in this, that $p \geq \frac{6k \ln n}{n}$ and $t > n/(2k)$ we get that

$$\begin{aligned} ne^{-p(t-1)/2} &< ne^{\frac{-pt}{2}+\frac{p}{2}} \\ &\leq ne^{\frac{-3\ln n}{2}+\frac{p}{2}} \\ &\leq nn^{-3/2}e^{p/2} \\ &\leq n^{-1/2}e^{1/2} = \sqrt{e}/\sqrt{n} \to 0 \end{aligned}$$

Now we show that the probability that the number of cycles of length k of less is greater than $n/2$, also goes to 0. Indeed, using Markov's inequality and the expected number of cycles we computed above, this probability is at most

$$(k-2)n^{k-1}p^k = (k-2)n^{k-1}n^{(\varepsilon-1)k} = (k-2)n^{k\varepsilon-1} < (k-2)n^{<0} \to 0.$$

As both probabilities go to 0, we have for large enough n that they are both less than 1/2, as needed.

□

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